Research Report

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Background

The Heisenberg Group is a $2n+1$ dimensional manifold denoted $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *, d_{cc})$, where $*$ is the group operation and d_{cc} is the Carnot–Carathéodory distance. This is a non commutative group with a natural anisotropic dilations $\delta_r(x,y,t) := (rx, ry, r^2 t)$ and whose balls are not convex (see Figure 1(a)).
The topological dimension of $\mathbb{H}^n$ is $2n+1$, while its Hausdorff dimension with respect to the Carnot-Caratheodory distance is $2n+2$; such dimensional difference leads to the existence of regular surfaces in the Heisenberg sense that are fractal in the Euclidean sense (see [3]).
The Lie algebra of the left invariant vector fields of $\mathbb{H}^n$ has a standard orthonormal basis $\{X_j = \partial_{x_j} - \frac{1}{2} y_j \partial_t,\ Y_j = \partial_{y_j} + \frac{1}{2} x_j \partial_t\}_{j=1,\dots,n}$, called *horizontal* because it spans the so-called *horizontal bundle*; it also holds the core property that $[X_j, Y_j] = \partial_t =: T$ for each $j = 1, \dots, n$, where T is called the *vertical* direction. This division between horizontal and vertical qualificates the Heisenberg Group as Carnot group of step 2. One can notice how the horizontal subbundle changes inclination at every point (see Figure 1(b)), allowing movement from each point to each point following only horizontal paths.
With a dual argument, one can associate at these vector fields the corresponding differential forms respectively: dx_j's and dy_j's for X_j's and Y_j's, and $\theta := dt - \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)$ for T. They also divide in *horizontal* and *vertical* as before.

My Research

My research concerns the study of currents on the Heisenberg Group $\mathbb{H}^n$, which naturally involves $\mathbb{H}$-regular surfaces and Rumin cohomology. These three aspects are deeply related but also wide enough to allow to study them separately.
The aim is to prove compactness theorems in the Heisenberg group for currents, with the final goal to solve a mass-minimality problem given a fixed boundary (generically known as Plateau Problem). The compactness theorem will need two preliminary results, a deformation theorem and a closure theorem.

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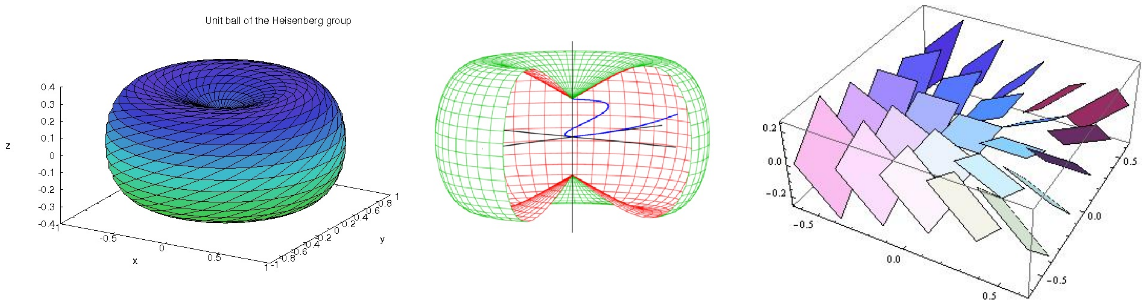

(a) Unit ball in $\mathbb{H}^1$ (b) Horizontal tangent space in $\mathbb{H}^1$

Figure 1: First Heisenberg Group $\mathbb{H}^1$.[1]

The strategy is to first build a grid in $\mathbb{H}^n$ whose faces are 1-codimensional $\mathbb{H}$-regular surfaces. Then create a deformation theorem to describe how a current can be approximated by currents supported on the grid. Finally prove a closure result for such approximations (see chapter 5 in [4]).

Rumin cohomology

A natural cohomology for the Heisenberg group is the Rumin cohomology, from which derives also the definition of currents (see [6] and 5.8 in [2]).
Take Ω^k as the set of all k-differential forms in $\mathbb{R}^k$ and conside the sets:

- $I^k = \{\alpha \wedge \theta + \beta \wedge d\theta \ / \ \alpha \in \Omega^{k-1}, \ \beta \in \Omega^{k-2}\}$
- $J^k = \{\alpha \in \Omega^k \ / \ \alpha \wedge \theta = 0, \ \alpha \wedge d\theta = 0\}$

The Rumin complex, due to Rumin in [6], is given by

$$0 \to \mathbb{R} \to C^\infty \overset{d_Q}{\to} \frac{\Omega^1}{I^1} \overset{d_Q}{\to} \cdots \overset{d_Q}{\to} \frac{\Omega^n}{I^n} \overset{D}{\to} J^{n+1} \overset{d_Q}{\to} \cdots \overset{d_Q}{\to} J^{2n+1} \to 0$$

where d is the standard differential operator and, for $k < n$, we have

$$d_Q([\alpha]_{I*}) := [d\alpha]_{I*},$$

while, for $k \geqslant n$,

$$d_Q := d_{|_{J*}}.$$

Finally, D is a 2^{nd}-order differential operator whose presence reflects the difference between the topological and Hausdorff dimensions of the space.

[1]pictures shown with permission of the authors.

Rumin forms are then defined as compactly supported on an open set U and their sets are denoted by

$$\mathcal{D}^k_{\mathbb{H}}(U) := \begin{cases} \frac{\Omega^k}{I^k} & , \text{ if } k \leqslant n \\ J^k & , \text{ if } k > n \end{cases}$$

As an example, in the first Heisenberg group $\mathbb{H}^1$ the Rumin complex is:

$$0 \to \mathbb{R} \overset{d_Q}{\to} \mathcal{D}^1_{\mathbb{H}}(U) \overset{D}{\to} \mathcal{D}^2_{\mathbb{H}}(U) \overset{d_Q}{\to} \mathcal{D}^3_{\mathbb{H}}(U) \to 0$$

with $\mathcal{D}^1_{\mathbb{H}}(U) = \text{span}\{dx, dy\}$, $\mathcal{D}^2_{\mathbb{H}}(U) = \text{span}\{dx \wedge \theta, dy \wedge \theta\}$ and $\mathcal{D}^3_{\mathbb{H}}(U) = \text{span}\{dx \wedge dy \wedge \theta\}$. When $k > 1$, examples becomes much more complicated.

While studying these differential operators, one can naturally ask whether they commute with the pullback of some functions. It turns out that the answer is positive.
Consider $f : \mathbb{H}^n \to \mathbb{H}^n$ and call if *contact map* iff

$$f_* \left(span\{X_1, \dots, X_n, Y_1, \dots, Y_n\} \right) \subseteq span\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$$

or, equivalently,

$$\iff f^*\theta = \lambda\theta.$$

Theorem 0.1. Consider a contact map $f : \mathbb{H}^n \to \mathbb{H}^n$. We have that

$$f^* d_Q = d_Q f^* \text{ for } k \neq n$$

and

$$f^* D = D f^* \text{ for } k = n.$$

Namely, the pullback of a contact map f commutes with the differental operators of the Rumin complex.

One can also notice that pushforward and pullback of contact forms in the Heisenberg group are quite interesting objects in their behaviour. It is lengthy but possible to write explicit pushforward and pullback formulas in the Rumin notation in $\mathbb{H}^1$.

Currents

Currents are linear functionals on differential forms and can be associated, under correct hypotheses, to surfaces. It is then possible to study minimality problems on them by measuring the mass of the associated surface. In the Euclidean space $\mathbb{R}^n$, Federer [1] used this association to prove the compactness theorem for currents, namely that the space of currents with compact support and with their mass and the mass of their boundaries uniformly bounded, is compact. This leads immediately to the existence of area-minimizing currents (indeed the solution to the Plateau Problem). My aim is to prove similar theorems for the Heisenberg Group $\mathbb{H}^n$, with its intrinsic geometrical constrains.

For $1 \leqslant k \leqslant n$, one can then define $\mathbb{H}-$ *k-dimensional currents* ($\mathcal{D}_{\mathbb{H},k}(U)$) as continuous linear functionals : $\mathcal{D}_{\mathbb{H}}^{k}(U) \to \mathbb{R}$, and $\mathbb{H}-$ *k-codimensional currents* ($\mathcal{D}_{\mathbb{H},2n+1-k}(U)$) as continuous linear functionals : $\mathcal{D}_{\mathbb{H}}^{2n+1-k}(U) \to \mathbb{R}$.
To associate these currents to surface, that we will call $\mathbb{H}$-regular, we need quite strong hypotheses. Before that, we can consider much less hypotheses and work with currents *representable by integration*. Call ${}_{H}\bigwedge_{k}$ the set of integrable k-vector (for precise definition see [2]). Then we can define:

Definition 0.2 (representable by integration)**.** $1 \leqslant k \leqslant 2n$. Let $S \in \mathcal{D}_{\mathbb{H},k}(U)$ and consider $U \in \mathbb{H}^n$ open.

$$S \text{ is } \textit{representable by integration} \iff$$

$\exists \mu_S$ a Radon measure over U and $\exists \overrightarrow{S} : U \to {}_{H}\bigwedge_{k}$ μ_S-measurable s.t. $\left\|\overrightarrow{S}(x)\right\| = 1$ for μ_S-a.a. $x \in U$ and

$$S(\omega) = \int \langle \omega(x) | \overrightarrow{S}(x) \rangle d\mu_S(x) \ \ \forall \omega \in \mathcal{D}_{\mathbb{H}}^{k}(U).$$

If S is such, we notationally write

$$S = \overrightarrow{S} \wedge \mu_S.$$

This definition is justified by an equivalence theorem, whose Riemannian version is due to Federer:

Theorem 0.3 (Equivalence Theorem)**.** Let $U \subseteq \mathbb{H}^n$ open. Let $S \in \mathcal{D}_{\mathbb{H},k}(U)$.

$$\|S\|(f) < \infty \ \ \forall f \in C^0_{\mathbb{H},0}(U)^+ \iff$$

$\exists \mu_S$ a Radon measure over U and $\exists \overrightarrow{S} : U \to {}_{H}\bigwedge_{m}$ μ_S-measurable s.t. $\left\|\overrightarrow{S}(x)\right\| = 1$ for μ_S-a.a. $x \in U$ and

$$S(\omega) = \int \langle \omega(x) | \overrightarrow{S}(x) \rangle d\mu_S(x) \ \ \forall \omega \in \mathcal{D}_{\mathbb{H}}^{k}(U),$$

where

$$Y = {}_{H}\bigwedge^{k}, \ \ C^0_{\mathbb{H},0}(U)^+ = \{f \in C^0_{\mathbb{H},0}(U) \ / \ f \geqslant 0\}$$

and

$$\|S\|(f) = \sup_{\varphi \in C^\infty(U,Y), \ \|\varphi\|_Y \leqslant f} S(\varphi).$$

As in the Riemannian case, it is possible to obtain some properties for such currents. Other properties, that depend more on the geometrical structure of the ambient set, are still under investigation. They are also related to "slicing theory" in the Heisenberg group, that will be needed for the closure result for the deformation theorem.

$\mathbb{H}$-regular surfaces

To associate currents to surfaces (we state here only the definition in the codimensional case), first call S a $\mathbb{H}$-*regular k-codimensional surface* in $\mathbb{H}^n$, $1 \leqslant k \leqslant n$, if and only if S is locally the level set of a $\left[C^1_{\mathbb{H}}\right]^k$-function $f : U \to \mathbb{R}^k$ with the horizontal gradient of each component $\nabla_{\mathbb{H}} f_i \neq 0$ for $i = 1, \ldots, k$. Then (proposition 5.15 in [2]) every such S, oriented by a tangent vector field $\overrightarrow{S}$, defines a $\mathbb{H}-k$-codimensional current $[\![S]\!]$ by $[\![S]\!](\omega) = \int_S \langle \overrightarrow{S} | \omega \rangle d\mathcal{S}^{2n+2-k}_{\infty}$, where $\omega \in \mathcal{D}^{2n+1-k}_{\mathbb{H}}(U)$ and $d\mathcal{S}^{2n+2-k}_{\infty}$ is a spherical Hausdorff measure. With these hypotheses, one can identify the surface S and the current $[\![S]\!]$. Finally, for all currents one can define a mass as $M(S) := \sup_{\|\omega_x\| \leqslant 1} S(\omega)$.

It is important to notice that the boundary of a current is also a current and that, in the first Heisenberg group $\mathbb{H}^1$, a 1-codimensional current $S : \mathcal{D}^2_{\mathbb{H}}(U) \to \mathbb{R}$ has a boundary $\partial S : \mathcal{D}^1_{\mathbb{H}}(U) \to \mathbb{R}$ that is not a codimensional current anymore, but instead a 1-dimensional current, by definition of k-currents, $1 \leqslant k \leqslant n$, and of the Rumin cohomology. Then, for S to be meaningful, ∂S will need to have only horizontal tangent vectors and this restricts considerably the amount of currents one can use.
On the other hand, in $\mathbb{H}^n$ with $n \geqslant 2$, a 1-codimensional current $S : \mathcal{D}^{2n}_{\mathbb{H}}(U) \to \mathbb{R}$ has a boundary $\partial S : \mathcal{D}^{2n-1}_{\mathbb{H}}(U) \to \mathbb{R}$ that is still codimensional (2-codimensional) and no horizontality is required.
Because of this issue, it seems very unlikely to be able to build a grid of 1-codimensional $\mathbb{H}$-regular surfaces in the case $\mathbb{H}^1$. For $k > n$, the problem is currently under study.

We know now how currents can be associated to $\mathbb{H}$-regular surfaces and that the $\mathbb{H}$-regularity is quite a strong condition, to the point that one should wonder whether these condition implies also $\mathbb{H}$-orientability (or orientability in the Heisenberg sense).
Consider now S a 1-codimensional C^1-Euclidean surface in $\mathbb{H}^1$, with $C(S) = \varnothing$ (the set of characteristic points). This implies that S is $\mathbb{H}$-regular. We can define:

Definition 0.4. S a $\mathbb{H}$ is $\mathbb{H}$-*orientable* if and only if $\exists$ a globally continuous horizontal vector field $W \neq 0$ on S that is $\mathbb{H}$-normal to S.

This notion is invariant for dilations and left-translations and it turns out that

Proposition 0.5. Let S be a 1-codimensional C^1-Euclidean surface in $\mathbb{H}^1$, with $C(S) = \varnothing$ and $\dim_{\mathcal{H}_E} S = 2$. If $S \in C^2_{\mathbb{H}}$, then

$$S \text{ is } \mathbb{H}\text{-orientable} \implies S \text{ is Euclidean-orientable} .$$

We can read this implication in the opposite direction and say that a non Euclidean surface who satisfies the hypotheses of the proposition is also a not-$\mathbb{H}$-orientable $\mathbb{H}$-regular surface. It turns out that there exists at least one such surface and so we conclude that not-$\mathbb{H}$-orientable $\mathbb{H}$-regular surfaces exist.

Then it will make sense to consider Heisenberg currents mod 2 (whose interpretation is to ignore the orientability (see, for instance, [5]). For such currents it could be possible to discuss a Plateau Problem as it is for the normal currents.

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