UNIVERSITY OF JYVÄSKYLÄ

UNIVERSITY OF TRENTO - Italy

MAnET Metric Analysis Meeting
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A new stable surface in the Heisenberg group

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Introduction

The Heisenberg group, $\mathbb{H}$: three dimensional, simply connected, nilpotent, non-Abelian Lie group. In exponential coordinates, we identify $\mathbb{H} = \mathbb{R}^3$.

- **Group operation**

$$(a, b, c) * (x, y, z) = \left(a + x, b + y, c + z + \frac{1}{2}(ay - bx)\right),$$

- **Left-invariant vector fields**

$$X(x, y, z) = \partial_x - \frac{1}{2} y \partial_z$$
$$Y(x, y, z) = \partial_y + \frac{1}{2} x \partial_z$$
$$Z(x, y, z) = \partial_z,$$

Introduction

- **Intrinsic graph** of $f : \mathbb{R}^2 \to \mathbb{R}$:

$$\begin{aligned}\Gamma_f &= \{(0,\eta,\tau) * (f(\eta,\tau),0,0) : (\eta,\tau) \in \mathbb{R}^2\} \\ &= \left\{(f,\eta,\tau - \frac{1}{2}\eta f) : (\eta,\tau) \in \mathbb{R}^2\right\},\end{aligned}$$

- **Intrinsic derivative** of f

$$\nabla^f f = \partial_\eta f + f\partial_\tau f,$$

which describes the tangent space of Γ_f,

- $\mathscr{C}^1_{\mathbb{W}}$ is the set of all **C^1-intrinsic functions**, i.e., $f : \mathbb{R}^2 \to \mathbb{R}$ such that both f and $\nabla^f f$ are continuous.
- **Intrinsic area** of Γ_f over $\omega \subset \mathbb{R}^2$

$$\int_\omega \sqrt{1 + (\nabla^f f)^2}\, \mathrm{d}\eta\, \mathrm{d}\tau.$$

Introduction

Remarks

- There is a natural notion of **subRiemannian perimeter** for subsets $E \subset \mathbb{H}$. If E has locally finite perimeter, then its reduced boundary $\partial^* E$ is "essentially" the intrinsic graph of a C^1-intrinsic function.

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- The set $\mathscr{C}^1_{\mathbb{W}}$ is NOT a vector space. Indeed,

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- By **minimal surface** in $\mathbb{H}$ we mean a topological surface that minimizes the area among all its bounded variations.

The problem

Bernstein's Problem: *Under which conditions on f is the following sentence true?*

If Γ_f is a minimal surface in $\mathbb{H}$, then Γ_f is a vertical plane.

We are interested in this problem because we want to better understand the space $\mathscr{C}^1_{\mathbb{W}}$ and the theory of perimeters in $\mathbb{H}$.

- If $f \in \mathscr{C}^1(\mathbb{R}^2)$, then it's true![1]
- If $f \in \mathscr{C}^0(\mathbb{R}^2)$ (even Lipschitz intrinsic), then it's false![2]

[1] M. Galli and M. Ritoré. "Area-stationary and stable surfaces of class C^1 in the sub-Riemannian Heisenberg group $\mathbb{H}^1$". In: Adv. Math. 285 (2015), pp. 737–765.

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- **What if $f \in \mathscr{C}^1_{\mathbb{W}}$?**

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The problem

Variational approach: if Γ_f is a minimal surface, then, for all $\psi \in \mathscr{C}_c^\infty(\omega)$, the following two conditions are satisfied:

1. $I_f(\psi) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_\omega \sqrt{1 + (\nabla^{f+\epsilon\psi}(f+\epsilon\psi))^2}\, \mathrm{d}\eta\, \mathrm{d}\tau \Big|_{\epsilon=0} = 0$
2. $II_f(\psi) = \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} \int_\omega \sqrt{1 + (\nabla^{f+\epsilon\psi}(f+\epsilon\psi))^2}\, \mathrm{d}\eta\, \mathrm{d}\tau \Big|_{\epsilon=0} \geq 0$

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This approach cannot work in general: If we only assume $f \in \mathscr{C}^1_{\mathbb{W}}$, then it is **not true** that the graph of the function $f + \epsilon\psi$ has necessarily locally finite area! Indeed,

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But this is another story...

We decided tackle the following problem:

If $f \in W^{1,1}_{loc}(\mathbb{R}^2) \cap \mathscr{C}^1_{\mathbb{W}}$ and Γ_f is a minimal surface in $\mathbb{H}$, then Γ_f is a vertical plane.

First Variation

For $f \in W^{1,1}_{loc}(\mathbb{R}^2) \cap \mathscr{C}^1_{\mathbb{W}}$ and $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^2)$,

$$I_f(\psi) = \int_{\mathbb{R}^2} \frac{\nabla^f f}{\sqrt{1 + (\nabla^f f)^2}} \left(\partial_\eta \psi + \partial_\tau (f\psi)\right) \mathrm{d}\eta \, \mathrm{d}\tau.$$

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If $f \in \mathscr{C}^2(\mathbb{R}^2)$, then

$$I_f(\psi) = - \int_{\mathbb{R}^2} \nabla^f \left(\frac{\nabla^f f}{\sqrt{1 + (\nabla^f f)^2}} \right) \psi \, \mathrm{d}\eta \, \mathrm{d}\tau.$$

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Thus, $I_f = 0$ if and only if

$$\nabla^f \nabla^f f = 0.$$

This is a second order differential equation.

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Thus, $I_f = 0$ if and only if

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This is a second order differential equation. However, we can interpret it as (*Lagrangian interpretation*)

$\nabla^f f$ is constant along the integral curves of the vector field
$\nabla^f = \partial_\eta + f\partial_\tau$.

First Variation

Conjecture

If $f \in W^{1,1}_{loc} \cap \mathscr{C}^1_{\mathbb{W}}$ and $I_f = 0$, then $\nabla^f f$ is constant along the integral lines of ∇^f, i.e., $\nabla^f \nabla^f f = 0$.

We are working on it.

However, we know that, *if $f \in W^{1,1}_{loc} \cap \mathscr{C}^1_{\mathbb{W}}$ and $\nabla^f f$ is constant along the integral lines of ∇^f, i.e., $\nabla^f \nabla^f f = 0$, then $I_f = 0$.*

$\nabla^f \nabla^f f = 0$

$$\nabla^f = \partial_\eta + f \partial_\tau$$

Lemma

Let $f \in \mathscr{C}^1_{\mathbb{W}}$. $\nabla^f f$ is constant along the integral curves of ∇^f if and only if there are continuous functions $A, B : \mathbb{R} \to \mathbb{R}$ such that

1. *all the integral curves of ∇^f are given by $t \mapsto (t, g(t, \zeta))$, where*

$$g(t, \zeta) = \frac{A(\zeta)}{2} t^2 + B(\zeta) t + \zeta;$$

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2. *The functions A, B must satisfy some properties. In particular, A must be non-decreasing;*
3. $f(t, g(t, \zeta)) = \partial_t g(t, z) = A(\zeta) t + B(\zeta);$
4. $\nabla^f f(t, g(t, \zeta)) = \partial_t^2 g(t, z) = A(\zeta).$

$\nabla^f \nabla^f f = 0$

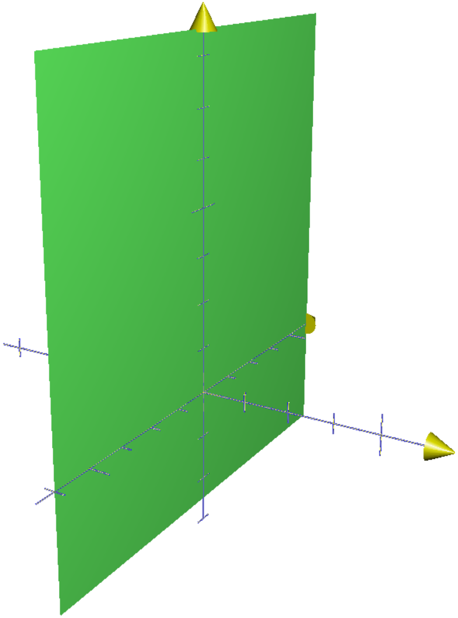
Consider the case ($B = 0$, $A : \mathbb{R} \to \mathbb{R}$ non-decreasing and continuous)

$$f\left(t, A(\zeta)\frac{t^2}{2} + \zeta\right) = A(\zeta)t.$$

We may construct the intrinsic graph of f in this way:

$$\Gamma_f = \{(0, 0, \zeta) + t(A(\zeta), 1, 0) : t, \zeta \in \mathbb{R}\}$$

$\nabla^f \nabla^f f = 0$: Examples



$$A(\zeta) = 0$$
$$f(\eta, \tau) = 0$$

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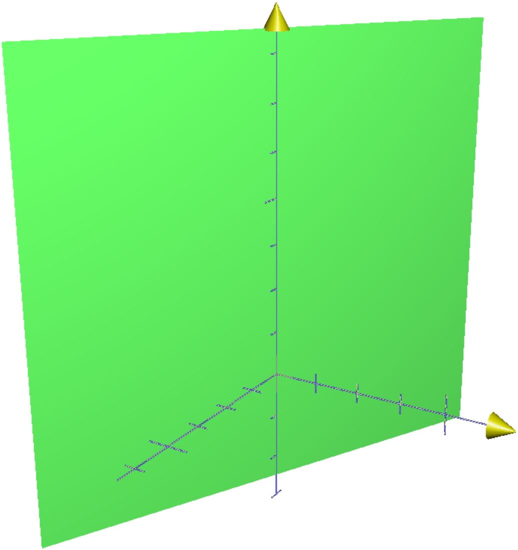

$$A(\zeta) = 1$$
$$f(\eta, \tau) = \eta$$

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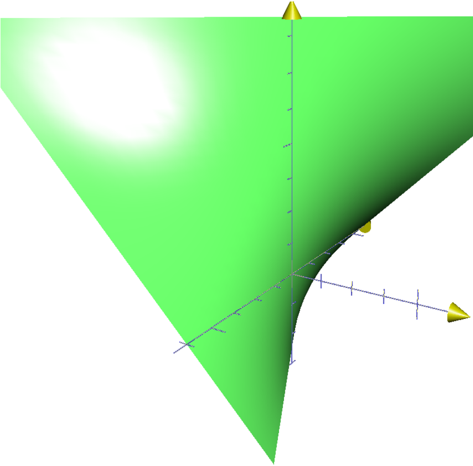

$$A(\zeta) = \zeta$$

$$f(\eta, \tau) = \frac{\eta}{\eta^2/2 + 1}$$

Second Variation

For $f \in W^{1,1}_{loc}(\mathbb{R}^2) \cap \mathscr{C}^1_{\mathbb{W}}$ and $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^2)$,

$$II_f(\psi) = \int_{\mathbb{R}^2} \frac{(\partial_\eta \psi + f\partial_\tau \psi + \psi \partial_\tau f)^2}{(1+(\nabla^f f)^2)^{3/2}} + \frac{\partial_\tau(\psi^2)\nabla^f f}{(1+(\nabla^f f)^2)^{1/2}} \, \mathrm{d}\eta \, \mathrm{d}\tau$$

Let's assume that f satisfies $\nabla^f \nabla^f f = 0$, with $A, B \in \mathscr{C}(\mathbb{R})$.

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Let's assume that f satisfies $\nabla^f \nabla^f f = 0$, with $A, B \in \mathscr{C}(\mathbb{R})$.

Proposition

If A (and B) is absolutely continuous, then

1. $f \in W^{1,1}_{loc}(\mathbb{R}^2) \cap \mathscr{C}^1_{\mathbb{W}}$;
2. $I_f = 0$;
3. *if $II_f \geq 0$, then $A' = B = 0$ and thus Γ_f is a vertical plane.*

A new stable surface

Theorem

Take $B = 0$ and

$$A(\zeta) = \begin{cases} 0 & \zeta \leq 0 \\ \textit{Cantor function} & 0 \leq \zeta \leq 1 \\ 1 & \zeta \geq 1 \end{cases}$$

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Take $B = 0$ *and*

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Then

1. $f \in W^{1,2}_{loc}(\mathbb{R}^2) \cap \mathscr{C}^1_{\mathbb{W}}$;

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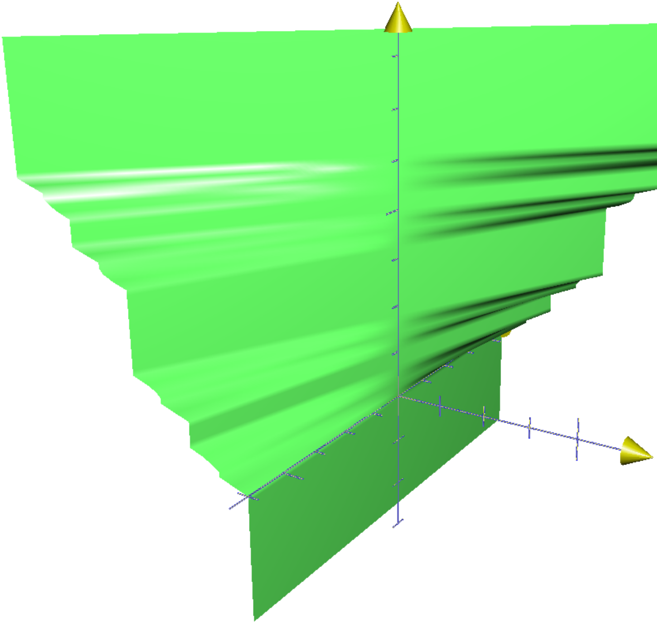
$\left.\begin{array}{l} I_f = 0; \\ II_f \geq 0. \end{array}\right\}$ Γ_f *is a stable surface*

However, we still don't know the answer to the question

Is Γ_f *a minimal surface?*

A new stable surface



Thank you for your attention!