Regularity of Quasilieaner equations in the Heisenberg Group

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Properties of Heisenberg Group

- $\mathbb{H}^n \cong \mathbb{R}^{2n+1}$, $x = (x_1, \ldots, x_{2n}, t), y = (y_1, \ldots, y_{2n}, s) \in \mathbb{H}^n$ the group law :

$$x * y = \left(x_1 + y_1, \ldots, x_{2n} + y_{2n}, t + s + \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i) \right)$$

- For $1 \leq i \leq n$, the Horizontal vector fields are given by

$$X_i := \partial_{x_i} - \frac{x_{n+i}}{2}\partial_t , \quad X_{n+i} := \partial_{x_{n+i}} + \frac{x_i}{2}\partial_t,$$

  The only non zero commutator $T := \partial_t = X_i X_{n+i} - X_{n+i} X_i$ is called the vertical vector field.
- The horizontal gradient : $\nabla_{\mathbb{H}} u = (X_1 u, \ldots, X_{2n} u)$ and the horizontal Hessian : $D^2_{\mathbb{H}} u = (X_i X_j u)_{i,j}$ for $u \in C^2$.
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$$d(x,0) \;\cong\; \|x\|_{\mathbb{H}^n} = \left(x_1^2 + \ldots x_{2n}^2 + |t|\right)^{\frac{1}{2}}$$

  and $d(x,y) = d(y^{-1} * x, 0)$ for all $x, y \in \mathbb{H}^n$.

- Hausdorff dimension of $(\mathbb{H}^n, d)$: $\dim_{\mathcal{H}}(\mathbb{H}^n) = Q := 2n+2$ w.r.t. this metric. Given any Korànyi ball $B(x,r)$ we have

$$|B(x,r)| \;=\; cr^Q$$

- Sobolev embedding : For $\Omega \subset \mathbb{H}^n$, horizontal Sobolev spaces are defined as $HW^{1,p}(\Omega) := \{u \in L^p(\Omega) \mid \nabla_{\mathbb{H}} u \in L^p(\Omega, \mathbb{R}^{2n})\}$. For all $u \in HW_0^{1,q}(B_r)$, there exists $c = c(n,q) > 0$ such that

$$\left(\int_{B_r} |u|^{\frac{Qq}{Q-q}}\, dx\right)^{\frac{Q-q}{Qq}} \;\leq\; c\left(\int_{B_r} |\nabla_{\mathbb{H}} u|^q\, dx\right)^{\frac{1}{q}},$$

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Quasilinear Equations of divergence form

Given $\Omega \subset \mathbb{H}^n$ and $A : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $B : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$, we consider the following equation

$$- \operatorname{div}_{\mathbb{H}} A(x, u, \nabla_{\mathbb{H}} u) \; + \; B(x, u, \nabla_{\mathbb{H}} u) \; = \; 0 \quad \text{in } \Omega$$

Regularity of horizontal gradient can be established by standard Morrey-Campanato type perterbation argument. Taking $B = 0$ without loss of generality, this involves freezing the coefficients as $\mathcal{A}(\nabla_{\mathbb{H}} u) = A(x_0, u(x_0), \nabla_{\mathbb{H}} u)$. This leads one to consider the following Dirichlet problem

$$\begin{cases} - \operatorname{div}_{\mathbb{H}}(\mathcal{A}(\nabla_{\mathbb{H}} u)) = 0 & \text{in } \Omega; \\ u = \phi & \text{in } \partial\Omega, \end{cases} \tag{1}$$

for some given $\phi : \overline{\Omega} \to \mathbb{R}$. In addition, we also assume the $2n \times 2n$ Jacobian $D\mathcal{A}(z) = (\partial \mathcal{A}_i(z)/\partial z_j)_{ij}$ to be *symmetric*.

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Isotropic growth condition

Structure condition

Given $g : C^1[0,\infty)$ satisfying $g(0) = 0$ and $\delta \leq tg'(t)/g(t) \leq g_0$

$$|\xi|^2 g(|z|)/|z| \;\leq\; \langle\, D\mathcal{A}(z)\,\xi, \xi\,\rangle \;\leq\; \Lambda\,|\xi|^2 g(|z|)/|z|$$
$$|\mathcal{A}(z)| \;\leq\; \Lambda\, g(|z|)$$

for every $z, \xi \in \mathbb{R}^{2n}$, where $\Lambda > 1$ and $g_0 > \delta \geq 0$.

Taking $G(t) = \int_0^t g(s)\,ds$, the model example is the minimization of the variational integral $I(u) = \int_\Omega G(|\nabla_{\mathbb{H}} u|)\,dx$, which leads to

$$-\;\mathrm{div}_{\mathbb{H}}\left[g(|\nabla_{\mathbb{H}} u|)\,\frac{\nabla_{\mathbb{H}} u}{|\nabla_{\mathbb{H}} u|}\right] \;=\; 0$$

In particular, $G(t) = t^p$ clearly, imples the sub p-laplace equation.

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Existence theory

- Monotonicity and Ellipticity : $\langle \mathcal{A}(z), z \rangle \geq \frac{1}{g_0} G(|z|)$.
- Existence of weak solution in Horizontal Orlicz-Sobolev space $HW^{1,G}(\Omega) = \{u \in L^G(\Omega) \mid \nabla_{\mathbb{H}} u \in L^G(\Omega, \mathbb{R}^{2n})\}$, using variational inequalitites of Kinderlehrer-Stampachhia.
- Comparison principle : u and v respectively are weak super and subsolution, if $u \geq v$ in $\partial\Omega$ then $u \geq v$ a.e. in Ω.

Theorem

Given a uniformly convex domain $D \subset \mathbb{R}^{2n+1}$ and $\phi \in C^2(\bar{D})$, if $u \in HW^{1,G}(D)$ is the weak solution of the Dirichlet problem

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Sketch of proof

The uniform convexity of the domain D, implies if $\nu(y)$ is the unit normal for every $y \in \partial D$, then $\langle\, x - y, \nu(y) \,\rangle \;\geq\; c|x-y|^2$ for all $x \in D$. This allows one to construct $\Gamma^{0,1}$ barrier functions

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since the Jacobian is *symmetric*. Hence $L^{-}(x) \;\leq\; u(x) \;\leq\; L^{+}(x)$ from comparison principle and the proof is finished by employing the maximum principle of differential ratio by Haar-Rado.

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- Ural'tseva (68), Evans (81): p-laplace equations, $p > 2$.
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For every $1 < p < \infty$ *and* $\varepsilon \geq 0$*, let* $u \in HW^{1,p}(\Omega)$ *be the weak solution of* $\mathrm{div}_{\mathbb{H}}\left[(\varepsilon + |\nabla_{\mathbb{H}} u|^2)^{\frac{p-2}{2}} \nabla_{\mathbb{H}} u\right] = 0$*, then* $|\nabla_{\mathbb{H}} u| \in L^\infty_{loc}(\Omega)$*. Moreover for all Korànyi balls* $B(x_0, 2r) \subset \Omega$

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Local boundedness of horizontal gradient

These pave the way for the result to be generalised in context of the general structure condition. The following is the first result.

Theorem (Local boundedness)

If $u \in HW^{1,G}(\Omega)$ be the weak solution of equation (1) with the given growth and ellipticity condition, then $\nabla_{\mathbb{H}} u \in L^\infty_{loc}(\Omega, \mathbb{R}^{2n})$ and there exists a constant $C = C(n, g_0, g(1), \Lambda) > 0$ such that the following holds a.e.

$$\sup_{B_{\sigma r}} G(|\nabla_{\mathbb{H}} u|) \leq \frac{C}{(1-\sigma)} \fint_{B_r} G(|\nabla_{\mathbb{H}} u|)\, dx$$

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Equation of derivatives

We denote the structure function $\mathrm{F}(t) = g(t)/t$. By appropriate regularization, one can assume $\varepsilon \leq \mathrm{F}(t) \leq 1/\varepsilon$ for a small $\varepsilon > 0$ (leading to the conclusion with $\varepsilon \to 0$). Thus, the global gradient bound together with the result of Capogna (97), implies

$$\nabla_{\mathbb{H}} u \in HW^{1,2}_{loc}(\Omega, \mathbb{R}^{2n}) \cap C^{0,\alpha}_{loc}(\Omega, \mathbb{R}^{2n}), \quad Tu \in HW^{1,2}_{loc}(\Omega) \cap C^{0,\alpha}_{loc}(\Omega)$$

This allows us to differentiate the equation $\operatorname{div}_{\mathbb{H}} \mathcal{A}(\nabla_{\mathbb{H}} u) = 0$ w.r.t. T (weakly), so that Tu is a weak solution of

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Using any $\lambda \geq 0$ and $\eta \in C_0^\infty(\Omega)$, we choose test function in the equations of derivatives. Choosing $\eta^2 G(|Tu|)^{\lambda+1} Tu$ in equation of Tu and using structure condition, we obtain

$$\int_\Omega \eta^2 G(|Tu|)^{\lambda+1} \mathrm{F}\left(|\nabla_{\mathbb{H}} u|\right) |\nabla_{\mathbb{H}}(Tu)|^2 \, dx \leq \frac{c}{(\lambda+1)^2} \int_\Omega |\nabla_{\mathbb{H}} \eta|^2 G(|Tu|)^{\lambda+1} \mathrm{F}\left(|\nabla_{\mathbb{H}} u|\right) |Tu|^2 \, dx$$

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$$\int_\Omega \eta^2 G(|\nabla_{\mathbb{H}} u|)^{\lambda+1} \mathrm{F}(|\nabla_{\mathbb{H}} u|) \, |D_{\mathbb{H}}^2 u|^2 \, dx \leq c \int_\Omega (|\nabla_{\mathbb{H}} \eta|^2 + \eta |T\eta|) G(|\nabla_{\mathbb{H}} u|)^{\lambda+2} \, dx + c(\lambda+1)^4 \int_\Omega \eta^2 G(|\nabla_{\mathbb{H}} u|)^{\lambda+1} \mathrm{F}(|\nabla_{\mathbb{H}} u|) \, |Tu|^2 \, dx$$

Reverse type inequality

The choice of $\eta^2 G(\eta)^{\lambda+1} G(|Tu|)^{\lambda+1} X_i u$ in equation of $X_i u$, along with the use of integral by parts and commutation relation, we have

$$\int_\Omega \eta^2 G(\eta)^{\lambda+1} G(|Tu|)^{\lambda+1} \mathrm{F}(|\nabla_{\mathbb{H}} u|)\, |D_{\mathbb{H}}^2 u|^2\, dx$$
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for all $\lambda \geq 1$, $\eta \geq 0$ and $K_\eta = \|\nabla_{\mathbb{H}} \eta\|^2_{L^\infty(\Omega)} + \|\eta T \eta\|_{L^\infty(\Omega)}$. The use of Young's inequality $st \leq \Psi(s) + \Psi^*(t)$ for conjugate pair (Ψ, Ψ^*) with $\Psi(s) = G(\sqrt{s})^{\lambda+1}$ allows one to remove Tu from the right. We can finally get

$$\int_\Omega \eta^2 G(|\nabla_{\mathbb{H}} u|)^{\lambda+1} \mathrm{F}(|\nabla_{\mathbb{H}} u|)\, |D_{\mathbb{H}}^2 u|^2\, dx \leq cK_\eta \int_{\mathrm{supp}(\eta)} G(|\nabla_{\mathbb{H}} u|)^{\lambda+2} dx$$

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Hölder continuity of horizontal gradient

The techniques developed also aids to establishing a $\Gamma^{1,\alpha}$ regularity of weak solutions of equation (1). By virtue of local boundedness theorem, we now denote

$$M_r \ := \ \max_{1\leq i\leq 2n} \sup_{B_r} |X_i u|$$

for every $B_r \subset\subset \Omega$. The next result is the following.

Theorem (Hölder continuity)

If $u \in HW^{1,G}(\Omega)$ is the weak solution of equation (1) then we have $\nabla_{\mathbb{H}} u \in C^{0,\beta/(1+g_0)}_{loc}(\Omega, \mathbb{R}^{2n})$ for some $\beta = \beta(n, g_0, g(1), \Lambda) \in (0,1)$. Moreover, if $B_{R_0} \subset\subset \Omega$, then for any $0 < R < R_0$ we have

$$\fint_{B_R} G(|\nabla_{\mathbb{H}} u - (\nabla_{\mathbb{H}} u)_{B_R}|)\, dx \ \leq \ C(n, g_0, g(1), \Lambda, M_{R_0}) \left(\frac{R}{R_0}\right)^{\beta}$$

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Integrability of Tu

The main tool that is required for the proof of Hölder continuity, is the higher integrability of the vertical derivative. This comes as a corrollary to the horizontal estimate established before, simply with the use of $|Tu| \leq 2|D^2_{\mathbb{H}} u|$. We obtain

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for any $\lambda \geq 1$, where $K_\eta = \|\nabla_{\mathbb{H}} \eta\|^2_{L^\infty(\Omega)} + \|\eta T\eta\|_{L^\infty(\Omega)}$. Thus for any fixed $q > Q$, one can choose large enough $\lambda >> 1$ such that the function $t^2 G(\sqrt{t})^{\lambda+1}$ dominates $t^{q/2}$. This makes the local integral $\int_{B_r} \mathrm{F}\left(|\nabla_{\mathbb{H}} u|\right) |Tu|^q \,dx$ to be bounded from above.

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Truncation of horizontal derivative

In order to deal with the weight $\mathrm{F}(|\nabla_{\mathbb{H}} u|)$ in the estimate, we use a double truncation of derivatives, that appears in Tolksdorff (82) and Lieberman (86). For some fixed $l \in \{1, \ldots, 2n\}$, we define $\mathrm{w} := \min\{M_R/4, (M_R/2 - X_l u)^+\}$, that is

$$\mathrm{w} = \begin{cases} M_R/4 & \text{if } X_l u \le M_R/4 \\ M_R/2 - X_l u & \text{if } M_R/4 \le X_l u \le M_R/2 \\ 0 & \text{if } M_R/2 < X_l u \end{cases}$$

Thus we have $0 \le \mathrm{w} \le M_R/4$ and $\{\mathrm{w} > 0\} \subseteq \{X_l u \le M_R/2\}$. Most importantly, $\nabla_{\mathbb{H}} \mathrm{w}$ vanishes outside $\{M_R/4 \le X_l u \le M_R/2\}$, and $\mathrm{F}(|\nabla_{\mathbb{H}} u|)$ is bounded from above and below, inside. Use of this truncation yields a Caccioppoli type estimate where the weight is dropped from both sides.

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The main lemma

We choose a standard test function $\zeta \in C_0^\infty(B_R)$ with $0 \leq \zeta \leq 1$ such that $\zeta \equiv 1$ in $B_{R/2}$, $|\nabla_{\mathbb{H}}\zeta| \leq 4/R$ and $|D^2_{\mathbb{H}}\zeta| \leq 16Q/R^2$. The following lemma is the key result to establish Hölder continuity of the horizontal gradient.

Lemma

Given the truncation w, *for every* $q > Q$ *and* $\beta \geq 2$, *we have the following Caccioppoli type estimate.*

$$\int_{B_R} \zeta^{\beta+2}\mathrm{w}^\beta |\nabla_{\mathbb{H}}\mathrm{w}|^2 dx \leq \frac{c\beta^4 M_R^4}{R^2}|B_R|^{\frac{2}{q}} \left(\int_{B_R} (\mathrm{w}\zeta)^{\frac{q}{q-2}(\beta-2)} dx\right)^{1-\frac{2}{q}}$$

This is clearly comparable to estimates corresponding to classical uniformly elliptic equations and the Hölder continuity of the horizontal gradient can be established using De Giorgi's methods.

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THANK YOU.