Currents in the Heisenberg Group, II part

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Last time...

Last time I talked about

- the deep motivation of my project: the Plateau (minimality) Problem
- and how it can be approached with the study of Currents

Sub-Riemannian Currents

Definition (Franchi et al. - 2007)

Let $U \subset \mathbb{H}^n$ open.
$\mathcal{D}^k_{\mathbb{H}}(U)$ = the set of *Heisenberg k-differentiable forms* := space of compactly supported smooth sections of ${}_H\bigwedge^k$.
${}_H\bigwedge^k$ = space of *integral k-covectors*.

Observation

If $n = 1$ and $k = 2$, then ${}_H\bigwedge^2 =< X \wedge T, Y \wedge T >$, with the usual notation of the Heisenberg Group.

Definition

$1 \leq k \leq 2n+1$. We call *Heisenberg k-current* any continuous linear functional on $\mathcal{D}^k_{\mathbb{H}}(U)$. We denote their set as $\mathcal{D}_{\mathbb{H},k}(U)$.

$\mathbb{H}$-regular surfaces on $\mathbb{H}^n$

Definition

$1 \leq k \leq n$. S is a k-dimensional $\mathbb{H}$-regular surface in $\mathbb{H}^n \iff$

$$\forall p \in S\ \exists U \subseteq \mathbb{H}^n \textit{ open},\ V \subseteq \mathbb{R}^k \textit{ open and } \varphi : V \to U \textit{ s.t. } p \in U,$$

$$\varphi\ 1-1,\ \textit{continuously } P-\textit{diff},\ d_H\varphi\ 1-1 \textit{ and } S \cap U = \varphi(V).$$

Definition

$n < k \leq 2n+1$. S is a k-dimensional $\mathbb{H}$-regular surface in $\mathbb{H}^n$ $\iff$

$$\forall p \in S\ \exists U \subseteq \mathbb{H}^n \textit{ open and } f : U \to \mathbb{R}^{2n+1-k} \textit{ s.t. } p \in U,$$

$$f \in [C^1_{\mathbb{H}}(U)]^{2n+1-k},\ \nabla_{\mathbb{H}} f \neq 0 \textit{ in } U \textit{ and } S \cap U = \{q \in U : f(q) = 0\}$$

ℍ-regular surfaces on $\mathbb{H}^n$

The cornerstone property of currents and ℍ-regular surfaces is 5.15 in *Franchi et al. - 2007*, part of whom is:

Proposition

Let $S \subseteq U$ be a k-dimensional ℍ-regular surface, oriented by a group tangent k-vector field $\overrightarrow{S}$. Then the map:

$$S : \omega \in \mathcal{D}^k_{\mathbb{H}}(U) \longmapsto \int_S \langle \overrightarrow{S}, \omega \rangle d\mathcal{S}^k_\infty \in \mathbb{R},\ if\ 1 \leq k \leq n$$

or

$$S : \omega \in \mathcal{D}^k_{\mathbb{H}}(U) \longmapsto \int_S \langle \overrightarrow{S}, \omega \rangle d\mathcal{S}^{k+1}_\infty \in \mathbb{R},\ if\ n < k \leq 2n+1$$

is a Heisenberg k-current and has locally finite mass.

$\mathbb{H}$-regular surfaces on $\mathbb{H}^n$

Definition

The *mass* M *of a current* S is

$$M(S) := \sup\{S(\alpha) \ / \ \alpha \in \mathcal{D}_{\mathbb{H}}^k(U), \|\alpha_x\| \leq 1 \ \forall x \in U\}$$

where

$$\|\alpha_x\| = \sup\{\alpha_x(\xi) \ / \ \xi \in \Lambda_k(U) \ \textit{simple}, \ \|\xi\| \leq 1\}.$$

Sets of Currents

Let $S \subseteq U$ be a k-dim $\mathbb{H}$-regular surface, oriented by a group tangent k-vector field $\overrightarrow{S}$. Following the last Proposition:

Definition (Rectifiable $\mathbb{H}$-Currents)

$S \in \mathcal{R}_{\mathbb{H},k}(U) \iff$ S can be written as

$$S(\varphi) = \int_S \langle \overrightarrow{S}, \varphi \rangle \mu(x) d\mathcal{S}^k_\infty, \; if \; k \leq n$$

or

$$S(\varphi) = \int_S \langle \overrightarrow{S}, \varphi \rangle \mu(x) d\mathcal{S}^{k+1}_\infty, \; if \; n < k \leq 2n+1,$$

where $\mu(x)$ is a positive integer multiplicity with finite integral.

Sets of Currents

Definition ($\mathbb{H}$-Currents Representable by Integration)

$S \in \mathcal{D}_{\mathbb{H},k}(U)$. S is *representable by integration*, $S = k \wedge \|S\|$,

$$\Longleftrightarrow$$

$\exists \|S\|$ a Radon measure over U and $\exists k : U \to_H \bigwedge_k$ $\|S\|$-measurable s.t. $\|k(x)\| = 1$ for $\|S\|$-a.a. $x \in U$ and

$$S(\omega) = \int_S \langle k(x), \omega(x) \rangle d\|S\|(x), \ \forall \omega \in \mathcal{D}_{\mathbb{H}}^k(U).$$

Remark

This is actually a theorem.

Sets of Currents

Observation

$S \in \mathcal{D}_{\mathbb{H},k}(U)$,

$$M(S) < \infty \Longrightarrow S = k \wedge \|S\|$$

$$M(S \llcorner A) = \|S\|(U \cap A)$$

Sets of Currents

Definition (Integral $\mathbb{H}$-Currents)

Let $U \subset \mathbb{H}^n$ open.
We call *integral k-dim* $\mathbb{H}$*-currents* the currents in the set

$$\mathcal{I}_{\mathbb{H},k}(U) := \{S \in \mathcal{R}_{\mathbb{H},k}(U) \ / \ \partial_H S \in \mathcal{R}_{\mathbb{H},k-1}(U)\}.$$

Definition (Normal $\mathbb{H}$-Currents)

Let $U \subset \mathbb{H}^n$ open.
We call *normal k-dim* $\mathbb{H}$*-currents* the currents in the set

$$\mathcal{N}_{\mathbb{H},k}(U) := \{S \in \mathcal{D}_{\mathbb{H},k}(U) \ / \ M(S) + M(\partial_H S) < \infty\}.$$

Sets of Currents

Definition (Boundary of a Current)

$S \in \mathcal{D}_{\mathbb{H},k}(U)$ and $\varphi \in \mathcal{D}_{\mathbb{H}}^{k-1}(U)$.

$$\partial_H S(\varphi) := S(d\varphi), \ \textit{if } k \neq n+1,$$

$$\partial_H S(\varphi) := S(D\varphi), \ \textit{if } k = n+1.$$

where D is the second order derivative of the Rumin complex.

Example

In $\mathbb{H}^1$ $\theta = dt + 2(xdy - ydx)$ and $D(\varphi) =$
$[\frac{1}{4}(XY\varphi_1 - XX\varphi_2) - T\varphi_1]dx \wedge \theta + [\frac{1}{4}(YY\varphi_1 - YX\varphi_2) - T\varphi_2]dy \wedge \theta.$

Slices on $\mathbb{H}^n$

Definition

Let $S \in \mathcal{N}_{\mathbb{H},k}(U)$, $u : \mathbb{H}^n \to \mathbb{R}$ Lipschitz and $r \in \mathbb{R}$.
We define a *slice of* S as

$$< S, u, r >:= \partial_H (S \llcorner \{u \leq r\}) - (\partial_H S) \llcorner \{u \leq r\} \ \in \mathcal{D}_{\mathbb{H},k-1}(U)$$

where

$$(S \llcorner A)(\varphi) := S(\mathcal{X}_A \varphi) \ , \ \varphi \in \mathcal{D}^k_{\mathbb{H}}(U).$$

Slices on $\mathbb{H}^n$

Proposition

Let $S \in \mathcal{N}_{\mathbb{H},k}(U)$, $u \in Lip(U, \mathbb{R})$, $r \in \mathbb{R}$. Then

- $(\|S\| + \|\partial_H S\|)(\{f = t\}) = 0$ for all but countably many t
- $< S, u, r >= (\partial_H S) \llcorner \{u \geq r\} - \partial_H (S \llcorner \{u \geq r\})$
- $spt < S, u, r > \subseteq f^{-1}\{r\} \cap sptS$
- $\partial_H < S, u, r >= - < \partial_H S, u, r >$

Orientability in $\mathbb{H}^1$

Let S be a C^1-Euclidean surface without characteristic points ($\nabla_H f \neq 0$ locally).

Observation

$\Longrightarrow$ S is a $\mathbb{H}$-regular surface.

Definition ($\mathbb{H}$-orientability)

S is $\mathbb{H}$-orientable $\Longleftrightarrow$ S has two lin. ind. continuous tangent vector fields:

- T
- $\alpha_1 X + \alpha_2 Y$, s.t. locally it is $\mu(Yf)X - \mu(Xf)Y$

$\Longleftrightarrow \exists W = W_1 X + W_2 Y$ continuous vector field s.t. locally $W = \lambda \nabla_H f$

Orientability in $\mathbb{H}^1$

Observation

$$\textit{Euclidean orientability} \Longrightarrow \mathbb{H}\textit{-orientability.}$$

If S is also $C^2_{\mathbb{H}}$:

$$\mathbb{H}\textit{-orientability} \Longrightarrow \textit{Euclidean orientability.}$$

Thank you!
Kiitos paljon!