Surface reconstruction in sub-Riemannian geometry

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Outline

- Motivation

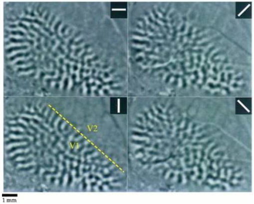
Part I

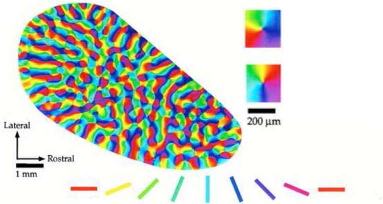
- Sub-Riemannian geometry
- Sub-Riemannian mean curvature flow
- Uniqueness in sub-Riemannian mean curvature flow

Part II

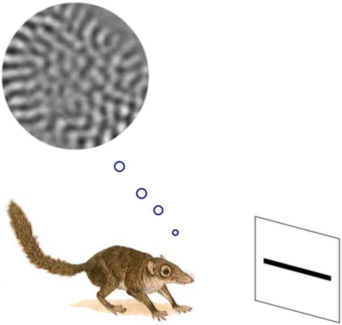
- Neurogeometry of primary visual cortex

Orientation preference map and pinwheels

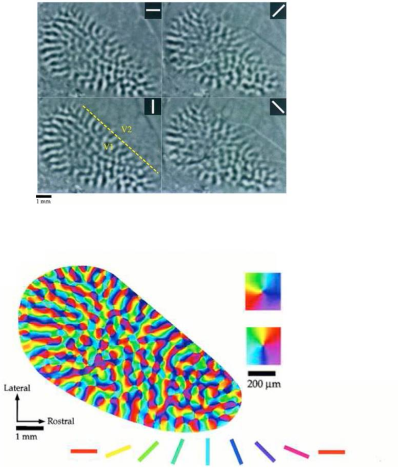




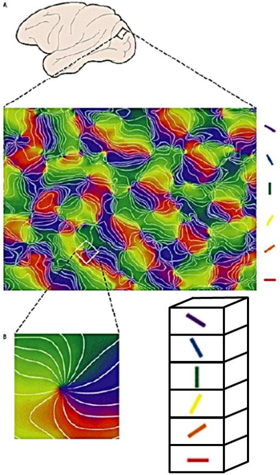
Orientation preference map
[Bosking et al., 1997]



Orientation preference map and pinwheels



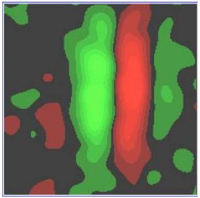
Orientation preference map
[Bosking et al., 1997]



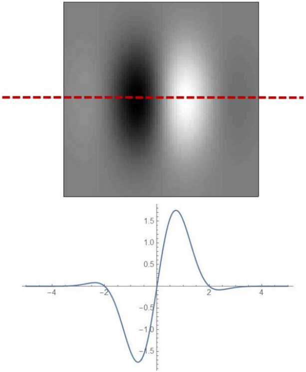
Pinwheels and hypercolumnar structure
[Petitot, *Neurogéométrie de la vision*, 2008]

Receptive profile

$$\xi = x\cos(\theta) + y\sin(\theta)$$
$$\eta = x\sin(\theta) + y\cos(\theta)$$
$$\Psi_0(\xi,\eta) = \mathrm{e}^{-\xi^2-\eta^2}\big(\cos(\eta) + i\sin(\eta)\big)$$



Experimental receptive profile



Gabor (real part) function

Set of receptive profiles: Fiber structure

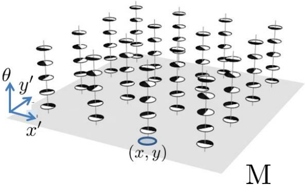
$$A_{x,y,\theta}(\xi,\eta) = \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\Psi_{x,y,\theta}(\tilde{\xi},\tilde{\eta}) = (A_{x,y,\theta}\Psi_0)(\tilde{\xi},\tilde{\eta}))$$

$$G_2 = \{A_\theta : \ \theta \in S^1\}$$

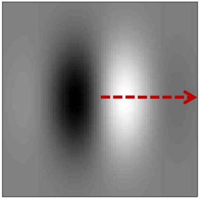
$$\mathrm{RP}_{(x,y)} = \{\Psi_{(x,y,\theta)} : \ (x,y) \quad \text{fixed}, \quad \theta \quad \text{variable}\}$$

$$= \{A_{(0,0,\theta)}\Psi_{(x,y,0)} : \ \theta \in S^1\}$$



Set of receptive profiles: 1-form

$$\xi = (\tilde{\xi} - x)\cos(\theta) + (\tilde{\eta} - y)\sin(\theta)$$
$$\eta = -(\tilde{\xi} - x)\sin(\theta) + (\tilde{\eta} - y)\cos(\theta)$$
$$\Psi(\xi,\eta) = \mathrm{e}^{-\xi^2+\eta^2}\big(\cos(\eta) + i\sin(\eta)\big)$$



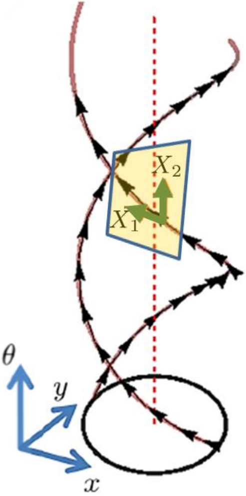
$$X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$

$$\boxed{\lambda = -\sin(\theta)d_x + \cos(\theta)d_y}$$

SE(2) geometry: Horizontality and neural connectivity

Horizontal tangent space:

$$\ker\lambda = \mathrm{span}\{X_1, X_2\}$$
$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y$$
$$X_2 = \partial_\theta$$

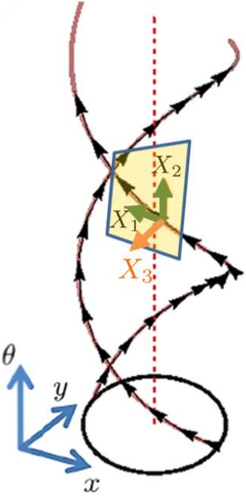


SE(2) geometry: Horizontality and neural connectivity

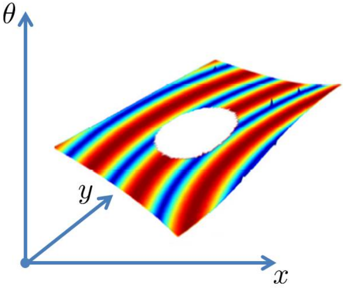
Horizontal tangent space:

$$\begin{aligned}
\ker\lambda &= \mathrm{span}\{X_1, X_2\} \\
X_1 &= \cos(\theta)\partial_x + \sin(\theta)\partial_y \\
X_2 &= \partial_\theta \\
[X_2, X_1] &= X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y
\end{aligned}$$

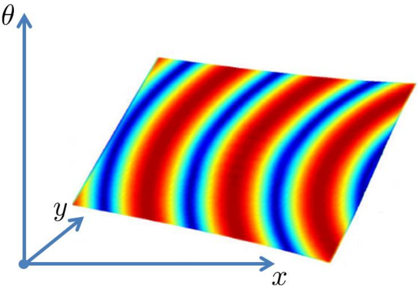
Hörmander condition!



Surface reconstruction



Corrupted surface



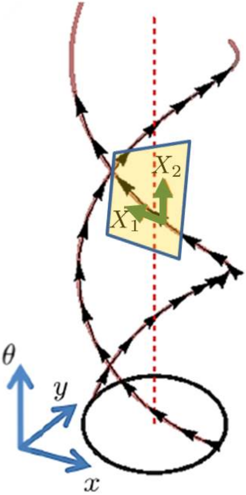
Reconstructed surface

PART I

Uniqueness in sub-Riemannian mean curvature flow

Preliminaries: SE(2) sub-Riemannian geometry

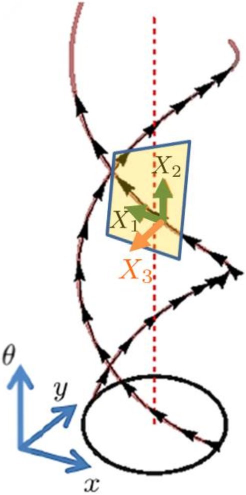
- Elements:
  $(x, y, \theta) \in \mathrm{SE}(2), \qquad x, y \in \mathbb{R}, \quad \theta \in S^1$
- Horizontal plane:
  $\mathrm{span}\{X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y,\ X_2 = \partial_\theta\}$
- For $u : \mathrm{SE}(2) \to \mathbb{R}$
  - horizontal gradient: $\nabla_h u = (X_1 u, X_2 u)$
  - horizontal divergence:
    $\mathrm{div}_h\, \nu = X_1\nu_1 + X_2\nu_2$
  - horizontal unit normal:
    $\nu_h = \frac{\nabla_h u}{|\nabla_h u|} = \frac{(X_1 u,\, X_2 u)}{\sqrt{(X_1 u)^2 + (X_2 u)^2}}$
  - horizontal Laplacian: $\Delta_h u = X_1^2 u + X_2^2 u$
  - horizontal mean curvature:
    $K_h = \mathrm{div}_h(\nu_h) = \mathrm{div}_h\left(\frac{\nabla_h u}{|\nabla_h u|}\right)$



Preliminaries: SE(2) sub-Riemannian geometry

$$X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$

- Elements:
  $(x, y, \theta) \in SE(2), \qquad x, y \in \mathbb{R}, \quad \theta \in S^1$
- Horizontal plane:
  $\text{span}\{X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y,\ X_2 = \partial_\theta\}$
- For $u : SE(2) \to \mathbb{R}$
  - full gradient: $\nabla u = (X_1 u, X_2 u, X_3 u)$
  - full divergence: $\text{div}\,\nu = X_1\nu_1 + X_2\nu_2 + X_3\nu_3$
  - full unit normal:
    $\nu = \frac{\nabla u}{|\nabla u|} = \frac{(X_1 u,\ X_2 u,\ X_3 u)}{\sqrt{(X_1 u)^2 + (X_2 u)^2 + (X_3 u)^2}}$
  - full Laplacian: $\Delta u = X_1^2 u + X_2^2 u + X_3^2 u$
  - full mean curvature:
    $K = \text{div}(\nu) = \text{div}\left(\frac{\nabla u}{|\nabla u|}\right)$



Preliminaries: SE(2) sub-Riemannian geometry

$$X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$

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Degenerate!

Preliminaries: SE(2) sub-Riemannian geometry

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  - full Laplacian: $\Delta u = X_1^2 u + X_2^2 u + X_3^2 u$
  - full mean curvature: $K = \text{div}(\nu) = \text{div}\left(\frac{\nabla u}{|\nabla u|}\right)$

Degenerate!

- Non-commutative Lie algebra:

$$[X_1,\ X_2] = -X_3 = \sin(\theta)\partial_x - \cos(\theta)\partial_y$$

Challenging but satisfies Hörmander condition!

Preliminaries: Sub-Riemannian mean curvature flow

$$\begin{cases} u_t = \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i u X_j u}{|\nabla_h u|^2} \right) X_{ij} u & \text{in } G \times (0, \infty) \\ u = u_0 & \text{on } G \times \{0\} \end{cases}$$

Characteristic points: $|\nabla_h u| = \sqrt{(X_1 u)^2 + (X_2 u)^2} = 0$

- Global description

  BUT...

- Not defined when $\nabla_h u = 0$!

Viscosity and vanishing viscosity

Regularized equation

$$\begin{cases} u_t^\epsilon = \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i u^\epsilon X_j u^\epsilon}{\epsilon^2 + |\nabla_h u^\epsilon|^2} \right) X_{ij} u^\epsilon \\ u^\epsilon(.,0) = u_0(.) \end{cases}$$

No characteristic points!

Degenerate equation

$$\begin{cases} v_t = \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i v X_j v}{|\nabla_h v|^2} \right) X_{ij} v \\ v(.,0) = v_0(.) \end{cases}$$

Not defined when $\nabla_h v = 0$!

$$\boxed{\begin{array}{c}\text{vanishing viscosity}\\ \lim\limits_{\epsilon \to 0} u^\epsilon\end{array}} = \boxed{\begin{array}{c}\text{viscosity}\\ v\end{array}}$$

Previously

- Heisenberg group, existence of graph, Capogna-Citti
- Heisenberg group, axisymmetricity, Ferrari-Liu-Manfredi

Difficulties due to characteristic points!

What about general setting?

Uniqueness in viscosity sense

Theorem (Uniqueness of viscosity solution)

Let v be a Lipschitz continuous viscosity solution to the degenerate problem and constantly 0 outside a compact set and let u^ϵ be the vanishing viscosity solution. Then for every $\alpha \in (0, \frac{1}{2})$ and $0 < T < \infty$ there exists a constant $M = M(u_0, T, \alpha)$ such that

$$\sup_{\xi \in SE(2), 0 \leq t \leq T} |(v - u^\epsilon)(\xi, t)| \leq M\epsilon^\alpha,$$

for all $0 < \epsilon < 1$.

Stability of vanishing viscosity

Unique viscosity solution implies

Corollary (Stability of vanishing viscosity)

For every $\alpha \in (0, \frac{1}{2})$ *and* $0 < T < \infty$ *there exists a constant* $M = M(u_0, T, \alpha)$ *such that*

$$\sup_{\xi \in SE(2), 0 \leq t \leq T} |(u^{\epsilon_1} - u^{\epsilon_2})(\xi, t)| \leq M(\epsilon_1 - \epsilon_2)^{\alpha},$$

for all $0 < \epsilon_1, \epsilon_2 < 1$ *and* $\epsilon_2 < \frac{\epsilon_1}{2}$.

Stability of vanishing viscosity

$$\sup_{\xi\in G, 0\le t\le T} \left|(u^{\epsilon_1} - u^{\epsilon_2})(\xi, t)\right| \le M(\epsilon_1 - \epsilon_2)^{\alpha}$$

$$\sup_{\xi\in G, 0\le t\le T} \left|(u^{\epsilon_1} - u^{\epsilon_2})(\xi, t)\right| \le M(\epsilon_1 - \epsilon_2)^{\alpha}$$

$$=$$

$$\sup_{\xi\in G, 0\le t\le T} \left|(u^{\epsilon_1} - u)(\xi, t)\right| \le M(\epsilon_1)^{\alpha} \quad \text{as} \quad \epsilon_2 \to 0$$

$$\Longrightarrow$$

$$u^{\epsilon_1} \to v \quad \text{as} \quad \epsilon_1 \to 0$$

Submitted to The Journal of Geometric Analysis

PART II

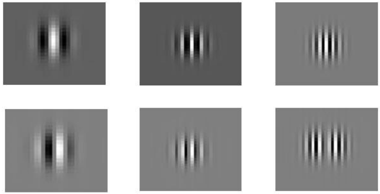
Neurogeometry of primary visual cortex V1

Extension: Orientation-frequency-phase selectivity

$$\xi = (\tilde{\xi} - x)\cos(\theta) + (\tilde{\eta} - y)\sin(\theta)$$

$$\eta = -(\tilde{\xi} - x)\sin(\theta) + (\tilde{\eta} - y)\cos(\theta)$$

$$\Psi_{x,y,\theta,\omega,\phi}(\xi,\eta) = \mathrm{e}^{-\xi^2+\eta^2}\big(\cos(\omega\eta + \phi) + i\,\sin(\omega\eta + \phi)\big)$$



$$\lambda = -\omega\sin(\theta)\partial_x + \omega\cos(\theta)\partial_y - d\phi$$

Extended geometry: Horizontality and neural connectivity

Extended 5-dim geometry

- Elements: $(x, y, \theta, \omega, \phi) \in \mathbb{R}^2 \times S^1 \times \mathbb{R} \times S^1$
- Horizontal tangent space:

$$
\begin{aligned}
X_1 &= \cos(\theta)\partial_x + \sin(\theta)\partial_y \\
X_2 &= \partial_\theta \\
X_3 &= -\sin(\theta)\partial_x + \cos(\theta)\partial_y + \omega\partial_\phi \\
X_4 &= \partial_\omega
\end{aligned}
$$

$$\ker \lambda = \mathrm{span}\{X_1, X_2, X_3, X_4\}$$

Extended geometry: Horizontality and neural connectivity

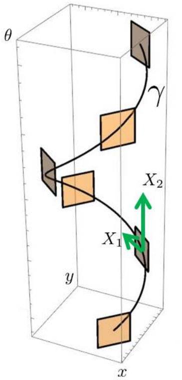
$$
\begin{aligned}
X_1 &= \cos(\theta)\partial_x + \sin(\theta)\partial_y \\
X_2 &= \partial_\theta \\
X_3 &= -\sin(\theta)\partial_x + \cos(\theta)\partial_y + \omega\partial_\phi \\
X_4 &= \partial_\omega \\
[X_3,\ X_4] &= -\partial_\phi \\
[X_1,\ X_2] &= \sin(\theta)\partial_x - \cos(\theta)\partial_y \\
[X_2,\ X_3] &= -\cos(\theta)\partial_x - \sin(\theta)\partial_y
\end{aligned}
$$

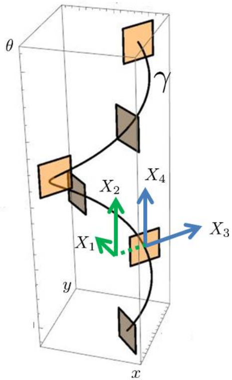
Extended geometry: Horizontality and neural connectivity

$$\begin{aligned}
X_1 =& \cos(\theta)\partial_x + \sin(\theta)\partial_y \\
X_2 =& \partial_\theta \\
X_3 =& -\sin(\theta)\partial_x + \cos(\theta)\partial_y + \omega\partial_\phi \\
X_4 =& \partial_\omega \\
[X_3,\ X_4] =& -\partial_\phi \\
[X_1,\ X_2] =& \sin(\theta)\partial_x - \cos(\theta)\partial_y \\
[X_2,\ X_3] =& -\cos(\theta)\partial_x - \sin(\theta)\partial_y
\end{aligned}$$

Hörmander condition!

Extended geometry: Tangent planes

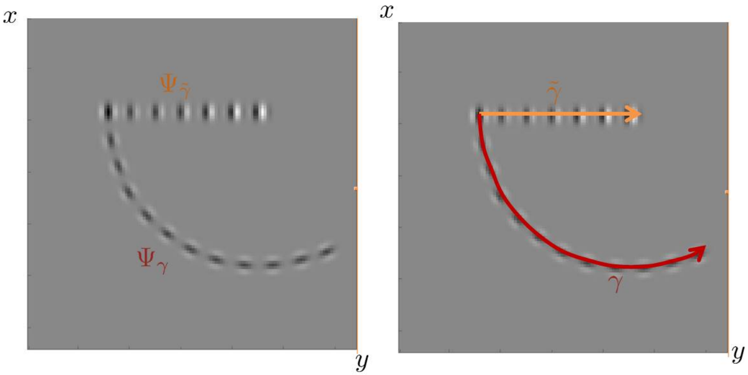




$$\gamma' = \gamma_1 X_1 + \gamma_2 X_2; \qquad \gamma_1,\ \gamma_2 \text{ constants}$$

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad X_2 = \partial_\theta$$

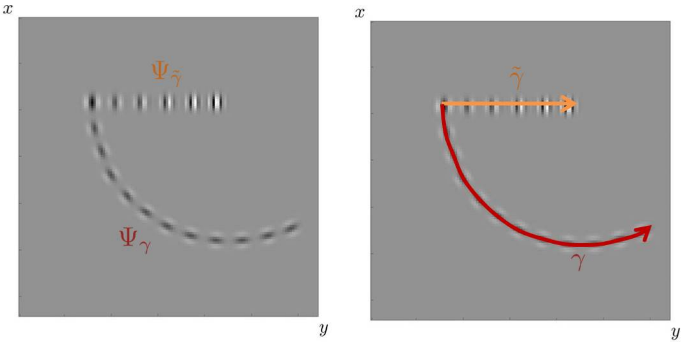
Extended geometry: Phase locked profiles



$$\gamma' = \gamma_1 X_1 + \gamma_2 X_2; \qquad \gamma_1,\ \gamma_2 \text{ constants}$$

$$\tilde{\gamma}' = \tilde{\gamma}_3 X_3; \qquad \tilde{\gamma}_3 \text{ constant}$$

Extended geometry: Phase locked profiles



$$\gamma' = \gamma_1 X_1 + \gamma_2 X_2; \qquad \gamma_1,\ \gamma_2 \text{ constants}$$

$$\tilde{\gamma}' = \tilde{\gamma}_3 X_3 + \tilde{\gamma}_4 X_4; \qquad \tilde{\gamma}_3,\ \tilde{\gamma}_4 \text{ constants}$$

Next step

- Comparison of simulation results with neurophysiological data
- Image completion via horizontal diffusion in the 5-dim extended geometry

Thank you!