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Geometric inequalities on the Heisenberg group

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MAnET Midterm Meeting,
Helsinki, 8-9 December 2015

...in the euclidean case

Brunn-Minkowski inequality

$|(1-s)A+sB|^{1/n} \geq (1-s)|A|^{1/n}+s|B|^{1/n}$ for all $A, B \subset \mathbb{R}^n$ Borel sets, $s \in [0,1]$

Prékopa-Leindler inequality

Let $f, g, h : \mathbb{R}^n \to [0,\infty)$ be measurable functions and fix $s \in (0,1)$.
$h((1-s)x+sy) \geq f(x)^{1-s} g(y)^s, \ \forall x, y \in \mathbb{R}^n \Rightarrow \int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-s} \left(\int_{\mathbb{R}^n} g\right)^s$

Borell-Brascamp-Lieb inequality

Let $f, g, h : \mathbb{R}^n \to [0,\infty)$ be measurable functions, fix $s \in (0,1)$ and $p \geq -\frac{1}{n}$.
$h((1-s)x+sy) \geq M_s^p(f(x), g(y)), \ \forall x, y \in \mathbb{R}^n \Rightarrow \int h \geq M_s^{\frac{p}{1+np}}\left(\int f, \int g\right)$,
where $M_s^p(a,b) = ((1-s)a^p + sb^p)^{1/p}$, for any $a, b > 0$, $p \in \mathbb{R} \setminus \{0\}$ and $s \in [0,1]$ and $M_s^0(a,b) = a^{1-s}b^s$, which is obtained from $M_s^p(a,b)$ by $p \to 0$.

Relation between the BM, PL and BBL inequalities

Observe that:

- Borell-Brascamp-Lieb inequality $\Rightarrow$ Prékopa-Leindler inequality
  As $M_s^0(a,b) = \lim_{p\to 0} M_s^p(s,b) = \lim_{p\to 0}((1-s)a^p + sb^p)^{\frac{1}{p}} = a^{1-s}b^s$, for all $a, b > 0$ and $s \in [0,1]$, PL can be obtained by setting $p = 0$ in BBL.
- Borell-Brascamp-Lieb inequality $\Rightarrow$ Brunn-Minkowski inequality
  Choosing f, g and h to be the characteristic functions of the Borel sets A, B, respectively $Z_s(A,B)$, these functions satisfy the condition of the BBL inequality, which implies that

$$|Z_s(A,B)|^{1/n} \geq (1-s)|A|^{1/n} + s|B|^{1/n}.$$

...in case of the Heisenberg group

How to define the intermediate points?

(Let $s \in [0, 1]$ be fixed.)

- in the euclidean case for the s-intermediate point associated to the pointpair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we use the convex combination $(1 - s)x + sy$

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- with the Heisenberg group operator $(*)$ and λ-dilation (ρ_λ) an s-intermediate point associated to the pointpair $(x, y) \in \mathbb{H}^n \times \mathbb{H}^n$ can be defined as $\rho_{1-s}(x) * \rho_s(y)$

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- with the help of geodesics an s-intermediate point between $x \in \mathbb{H}^n$ and $y \in \mathbb{H}^n$ can be defined as that point on the geodesic connecting the two points, which divides the geodesic in segments with ratio $s : (1 - s)$

... in case of the Heisenberg group

- When $y \in cut(x)$, the geodesic from x to y is not uniquely defined.
- Let's introduce the notation $Z_s(x,y)$ for the set of s-intermediate points associated to $(x,y) \in \mathbb{H}^n \times \mathbb{H}^n$:

$$Z_s(x,y) = \{z \in \mathbb{H}^n | d(x,z) = sd(x,y) \text{ and } d(z,y) = (1-s)d(x,y)\}$$

- For $A, B \subset \mathbb{H}^n$ define

$$Z_s(A,B) = \bigcup_{(x,y)\in A\times B} Z_s(x,y)$$

...in case of the Heisenberg group

Brunn-Minkowski inequality

$|Z_s(A,B)|^{1/d} \geq (1-s)|A|^{1/d} + s|B|^{1/d}$ for all $A, B \subset \mathbb{H}^n$ Borel sets, $s \in [0,1]$

Prékopa-Leindler inequality

Let $f, g, h : \mathbb{R}^n \to [0,\infty)$ be measurable functions and fix $s \in (0,1)$.

$$h(z) \geq f(x)^{1-s} g(y)^s,\ \forall x, y \in \mathbb{H}^n, z \in Z_s(x,y) \Rightarrow \int_{\mathbb{H}^n} h \geq \left(\int_{\mathbb{H}^n} f\right)^{1-s} \left(\int_{\mathbb{H}^n} g\right)^s$$

Borell-Brascamp-Lieb inequality

Let $f, g, h : \mathbb{H}^n \to [0,\infty)$ be measurable functions, fix $s \in (0,1)$ and $p \geq -\frac{1}{d}$. $h(z) \geq M_s^p(f(x), g(y)), \forall x, y \in \mathbb{H}^n, z \in Z_s(x,y) \Rightarrow$

$$\Rightarrow \int_{\mathbb{H}^n} h \geq M_s^{\frac{p}{1+dp}} \left(\int_{\mathbb{H}^n} f, \int_{\mathbb{H}^n} g\right)$$

... in case of the Heisenberg group

How to choose d?

Use for d

- the topological demension $2n+1$?
- the homogenous dimension $2n+2$?
- something else?

Sketch

- Proof for normalized functions f, g and $p = -\frac{1}{n}$.

Borell-Brascamp-Lieb inequality for normalized functions

Let $f, g, h : \mathbb{R}^n \to [0, \infty)$ be measurable functions with $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1$. Fix $s \in (0, 1)$.

$$h((1-s)x + sy) \geq M_s^{-\frac{1}{n}}(f(x), g(y)), \forall x, y \in \mathbb{R}^n \Rightarrow \int_{\mathbb{R}^n} h \geq 1$$

- Rescaling argument.

- Notation: $supp(f) = X, \; supp(g) = Y, \; \mu = f\mathrm{d}x, \; \nu = g\mathrm{d}y.$
- Consider a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for $S = \nabla\varphi : X \to Y$, $S\#\mu = \nu$.
- Consider the displacement interpolant measure of μ and ν: $[\mu, \nu]_s = (S_s)\#\mu$ with probability density ρ_s, where $S_s = (1-s)Id + s\nabla\varphi$.
- By the concavity of the $det(\cdot)^{1/n}$ function over symmetric, positive-semidefinite matrices

$$det((1-s)I_n + sHess(\varphi(x)))^{1/n} \geq (1-s)(det(I_n))^{1/n} + s(Hess(\varphi(x)))^{1/n}.$$

- Monge-Ampère for f and ρ_s: $f(x) = \rho_s(S_s(x))\mathrm{Jac}(S_s)(x), \; \mu - a.e \; x$, where $\mathrm{Jac}(S_s)(x) = det((1-s)I_n + sHess\varphi(x))$.
- Monge-Ampère for f and g: $f(x) = g(S(x))\mathrm{Jac}(S)(x), \; \mu - a.e \; x$, where $\mathrm{Jac}(S)(x) = det(Hess(\varphi(x)))$.

- $det((1-s)I_n + sHess(\varphi(x)))^{1/n} \geq (1-s)(det(I_n))^{1/n} + s(Hess(\varphi(x)))^{1/n}$
- $f(x) = \rho_s(S_s(x))det((1-s)I_n + sHess\varphi(x))$
- $f(x) = g(S(x))det(Hess(\varphi(x)))$

- $det((1-s)I_n + sHess(\varphi(x)))^{1/n} \geq (1-s)(det(I_n))^{1/n} + s(Hess(\varphi(x)))^{1/n}$
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- $f(x) = g(S(x))det(Hess(\varphi(x)))$

$$\begin{aligned}
\Rightarrow \left(\frac{f(x)}{\rho_s(x)}\right)^{1/n} &\geq (1-s) + s\left(\frac{f(x)}{g(S(x))}\right)^{1/n} \\
(\rho_s(x))^{-1/n} &\geq (1-s)(f(x))^{-1/n} + sg(S(x))^{-1/n} \\
\rho_s(x) &\leq M_s^{-1/n}(f(x), g(S(x)))
\end{aligned}$$

- $det((1-s)I_n + sHess(\varphi(x)))^{1/n} \geq (1-s)(det(I_n))^{1/n} + s(Hess(\varphi(x)))^{1/n}$
- $f(x) = \rho_s(S_s(x))det((1-s)I_n + sHess\varphi(x))$
- $f(x) = g(S(x))det(Hess(\varphi(x)))$

$$
\begin{aligned}
\Rightarrow \left(\frac{f(x)}{\rho_s(x)}\right)^{1/n} &\geq (1-s) + s\left(\frac{f(x)}{g(S(x))}\right)^{1/n} \\
(\rho_s(x))^{-1/n} &\geq (1-s)(f(x))^{-1/n} + sg(S(x))^{-1/n} \\
\rho_s(x) &\leq M_s^{-1/n}(f(x), g(S(x))) \leq h(Z_s(x, S(x)))\ \mu - a.e\ x
\end{aligned}
$$

$$\Downarrow$$

$$\int_{\mathbb{R}^n} h \geq 1$$

- On the Riemannian manifold M, the optimal transport map pushing μ forward to ν can be written, as $S(x) = \exp_x(-\nabla\varphi(x))$, where φ is a c-concave function relative to the support of f and g.
- S_s can be defined in point x as $\exp_x(-s\nabla\varphi(x))$.
- The Jacobian determinant of S_s satisfies the following inequality for $\mu - a.e.\ x$

$$(\mathrm{Jac}(S_s)(x))^{\frac{1}{n}} \geq (1-s)\left(v_{1-s}(S(x),x)\right)^{\frac{1}{n}} + s\left(v_t(x,S(x))\right)^{\frac{1}{n}}(\mathrm{Jac}(S)(x))^{\frac{1}{n}},$$

where the appearing volume distortion coefficient is defined as

$$v_s(x,y) = \lim_{r\to 0}\frac{vol(Z_t(x,B(y,r))}{vol(B(y,sr))},\ \forall x,y\in M \text{ and } s\in(0,1].$$

Borell-Brascamp-Lieb inequality on the Riemannian manifolds

Let M be a continuously curved, n-dimensional Riemannian manifold. Fix $p \geq -\frac{1}{n}$ and $s \in (0,1)$. Let $f, g, h : \mathbb{M} \to [0,\infty)$ be three measurable functions, A and B two Borel sets of M such that $suppf \subset A, suppg \subset B$ and assume that

$$h(z) \geq M_s^p \left(\frac{f(x)}{v_{1-s}(y,x)}, \frac{g(y)}{v_s(x,y)} \right) \quad \textit{for all } (x,y) \in A \times B, z \in Z_s(x,y).$$

Then

$$\int_M h \geq M_s^{\frac{p}{1+np}} \left(\int_M f, \int_M g \right).$$

N. Juillet disproves the existence of

- geodesic Brunn-Minkowski inequality in any dimension.
- mutiplicative Brunn-Minkowski for any dimension less than the topogical dimension of the Heisenberg group.

Bad news:

- In the Heisenberg group the optimal transport map doesn't assure the existence of such a determinant inequality that we used for the proof in the euclidean case and in the riemannian case, therefore to prove the Broell-Brascamp-Lieb inequality the same approach cannot be used.
- Juillet's disproval

Good news:

- We know from D. Cordero-Erasquin, R. J. McCann, M. Schmuckenschläger, that on the Riemannian manifolds a weighted Borell-Brascamp-Lieb inequality holds.
- L. Ambrosio and S. Rigot consider a Riemannian approximation for the Heisenberg group from the point of view of the optimal mass transportation approach.
- Using these two results we try to formulate a Borell-Brascap-Lieb inequality on the Heisenberg group, which can be obtained as a limit Riemannian Borell-Brascap-Lieb inequalities.

References I

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Thank you!