Sub-Riemannian Problems on Lie Groups with Applications to Medical Image Processing

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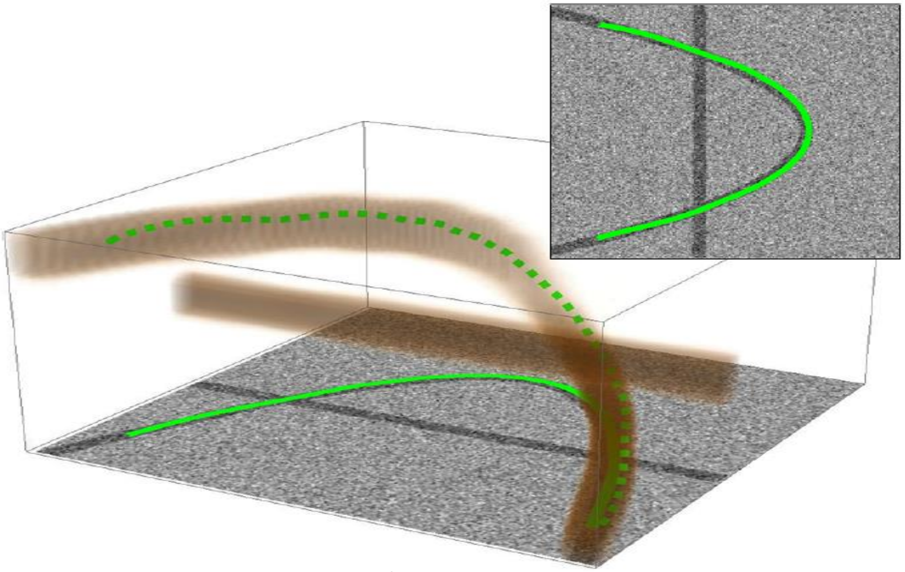
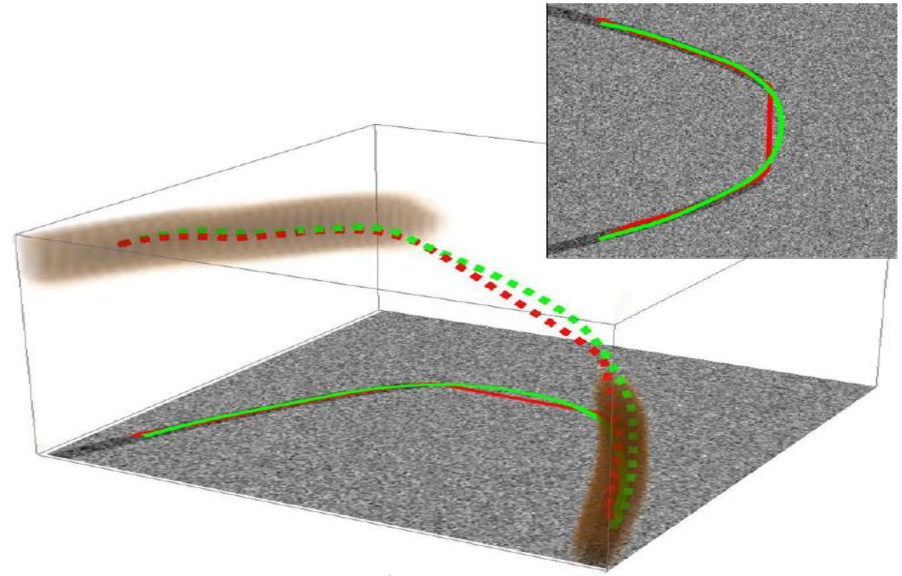
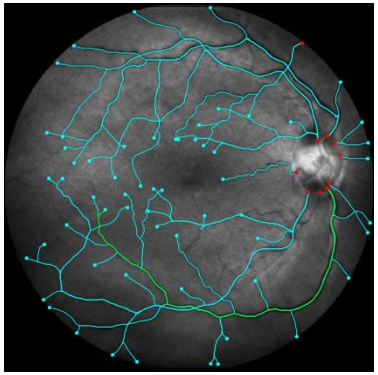
SR geodesics on Lie Groups in Image Analysis



Crossing structures are disentangled

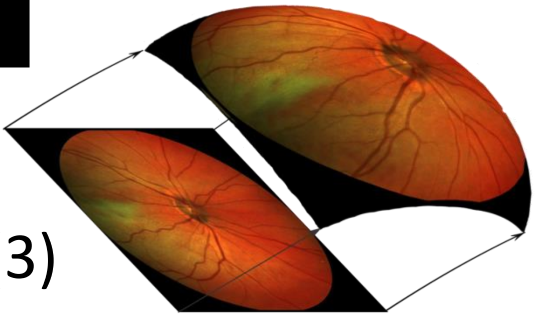


Restoration of corrupted contours based on model of human vision

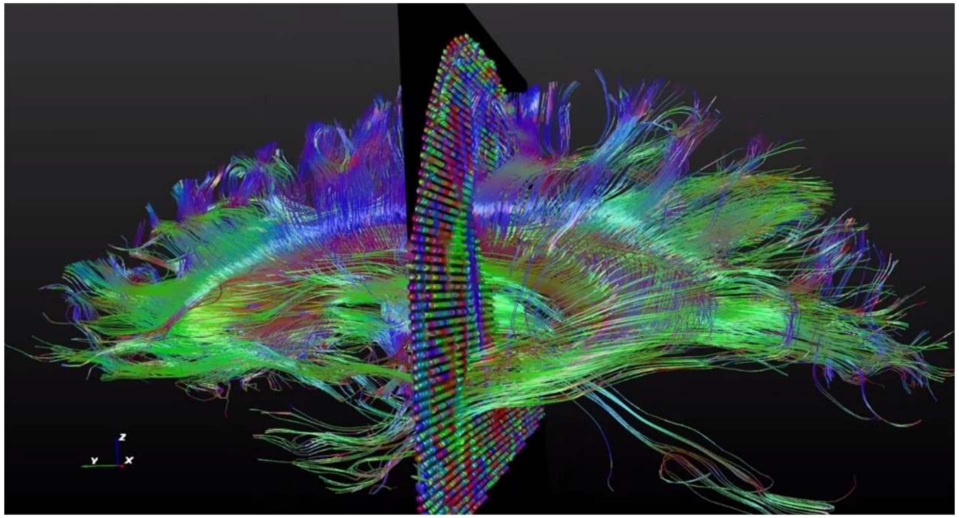


SE(2)

Applications in medical image analysis



SO(3)



SE(3)

Analysis of Images of the Retina

Diabetic retinopathy --- one of the main causes of blindness.
Epidemic forms: 10% people in China suffer from DR.
Patients are found early --> treatment is well possible.
Early warning --- leakage and malformation of blood vessels.
The retina --- excellent view on the microvasculature of the brain.

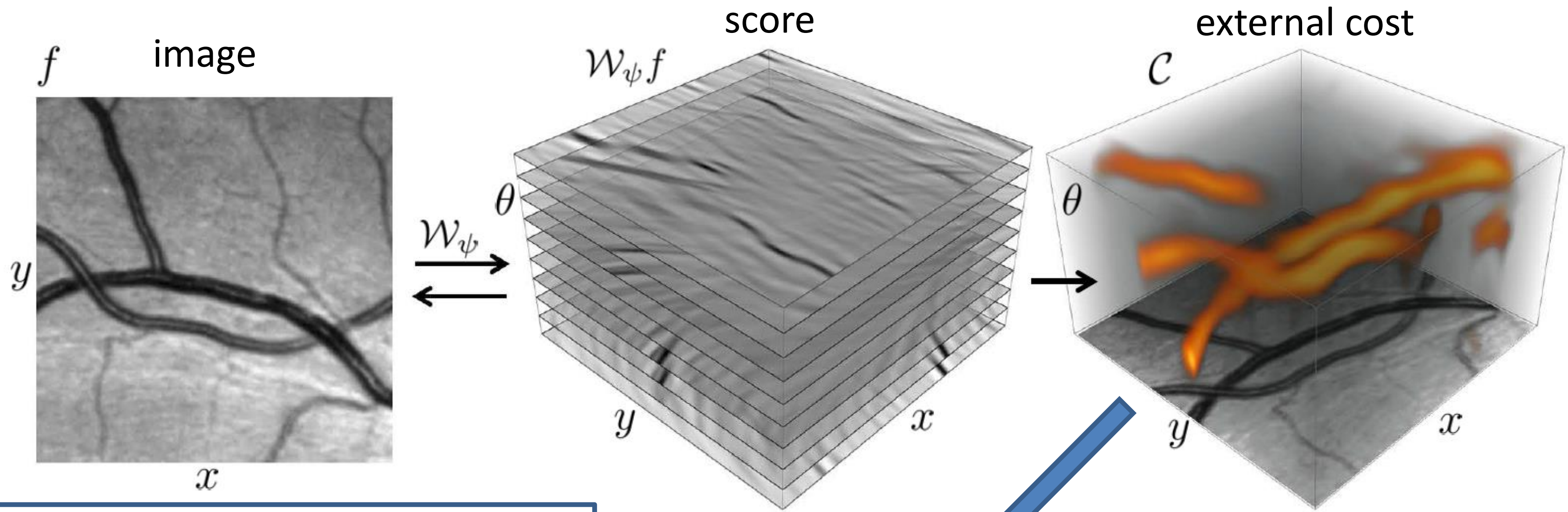
Healthy retina

Diabetes Retinopathy with tortuous vessels

Sub-Riemannian problem on SE(2) with given external cost

(with E.J. Bekkers, R. Duits and G. Sanguinetti)

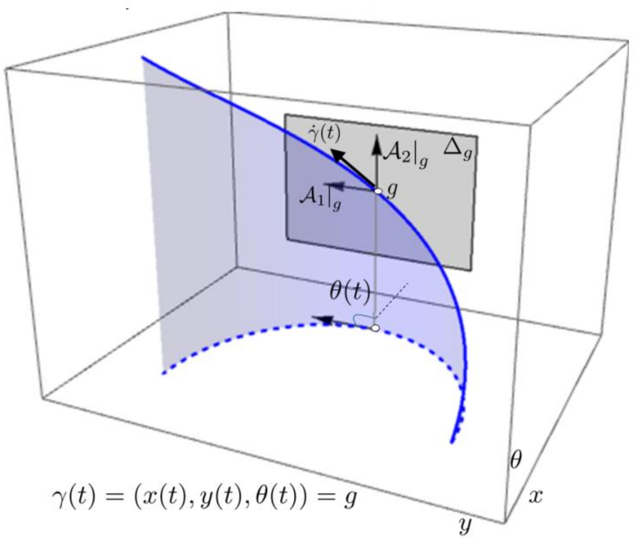
Data-driven Sub-Riemannian Geodesics in SE(2)



$$\mathcal{A}_1|_g = \cos\theta\ \partial_x|_g + \sin\theta\ \partial_y|_g$$

$$\mathcal{A}_3|_g = -\sin\theta\ \partial_x|_g + \cos\theta\ \partial_y|_g$$

$$\mathcal{A}_2|_g = \partial_\theta|_g$$



$$l = \int_0^{t_1} \mathcal{C}(\gamma(t)) \sqrt{\xi^2 u_1^2(t) + u_2^2(t)}\, dt \to \min,$$

$$\dot\gamma = u^1\ \mathcal{A}_1|_\gamma + u^2\ \mathcal{A}_2|_\gamma\,,$$

$$\gamma(0) = \mathrm{Id}, \qquad \gamma(T) = g,$$

$$\gamma(t) \in \mathrm{SE}(2),\ (u^1(t), u^2(t)) \in \mathbb{R}^2,\ \xi > 0.$$

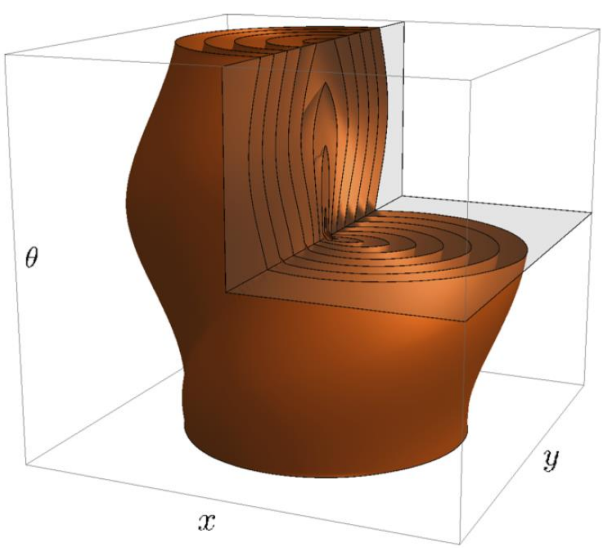
PDE-based Approach

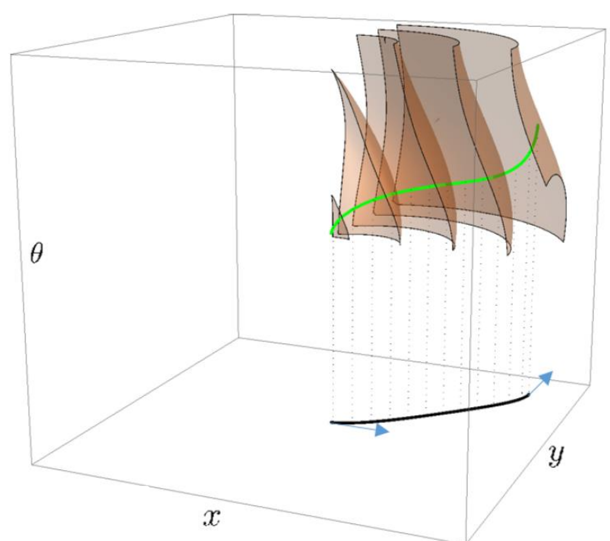
1. HJB equation for wavefront propagation

$$\begin{cases} \text{For } n = 1, 2, \ldots, \text{ and } r \in [n\epsilon, (n+1)\epsilon] : \\ \frac{\partial W^{\epsilon}_{n+1}}{\partial r}(g, r) = 1 - \mathcal{C}^{-1}(g)\sqrt{\left(\xi^{-2}|\mathcal{A}_1 W^{\epsilon}_{n+1}(g,r)|^2 + |\mathcal{A}_2 W^{\epsilon}_{n+1}(g,r)|^2\right)}, \\ W^{\epsilon}_{n+1}(g, n\epsilon) = W^{\epsilon}_{n}(g, n\epsilon), \text{ for } g \neq e, \\ W^{\epsilon}_{n+1}(e, n\epsilon) = 0, \\ \text{Initialization } (n = 0)\text{: } W^{\epsilon}_{1}(g, 0) = \delta^{M}_{e}(g) \end{cases}$$

2. Distance map $d(e, g) = \lim_{\epsilon \to 0} \lim_{n \to \infty} W^{\epsilon}_{n+1}(g, (n+1)\epsilon)$

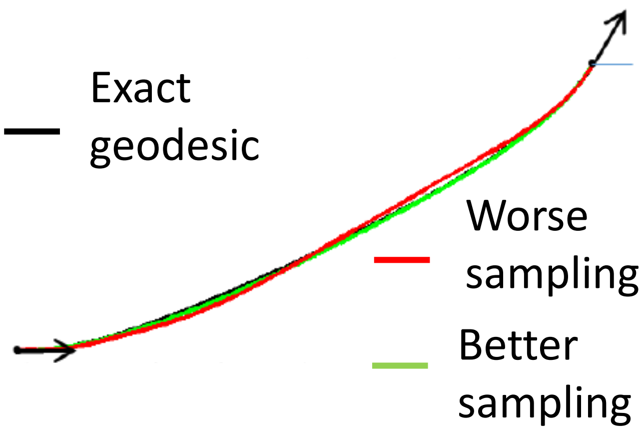
3. Minimizers by backward integration of the Hamiltonian system



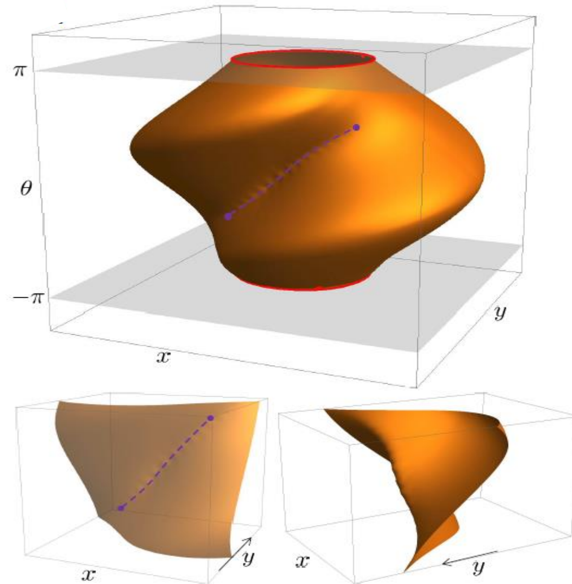


Numerical Verification for C=1

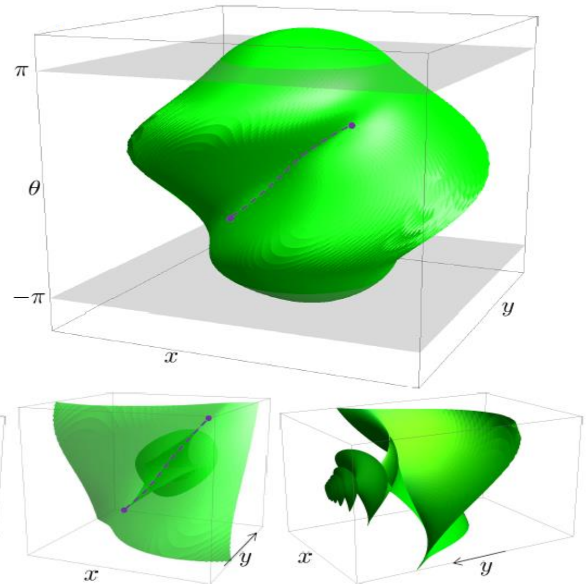
Converging to Exact Geodesics



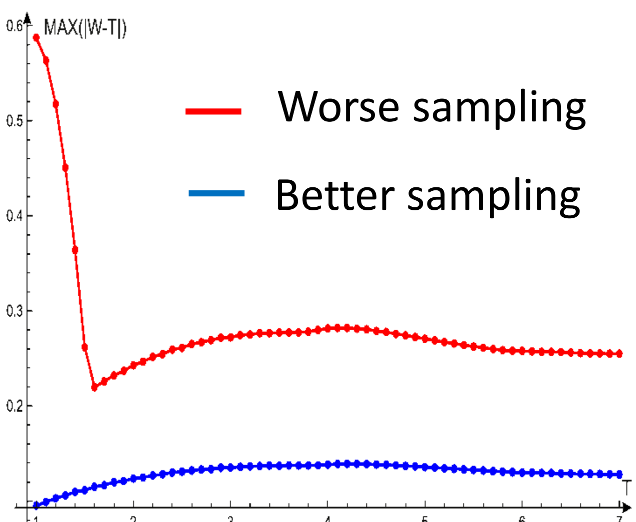
SR-sphere Numerically



Exact Wavefront

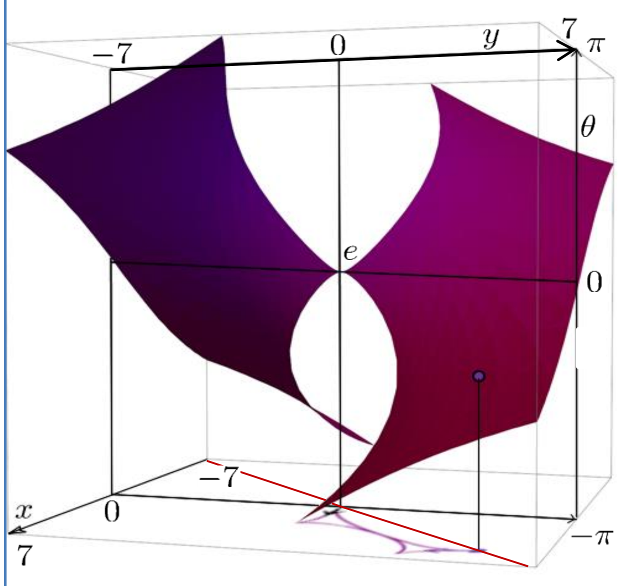


Max Absolute Error

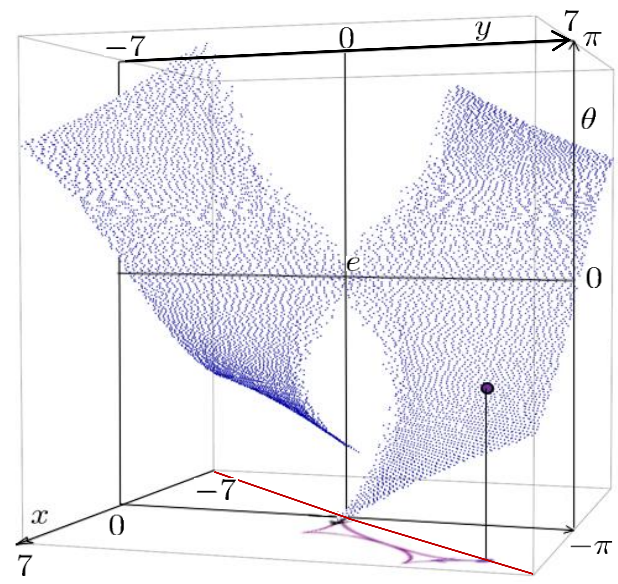


(T – radius of SR-sphere)

Maxwell Set Exact



Maxwell Set Numerically

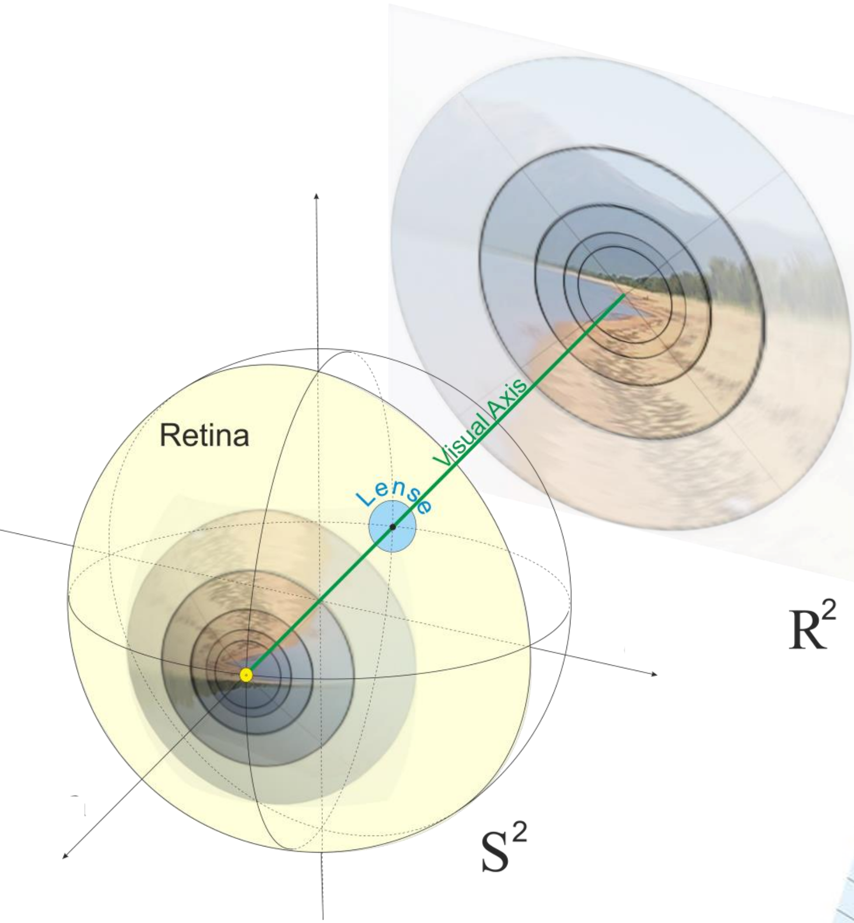


Sub-Riemannian problem on SO(3) with cuspless spherical projection constraint

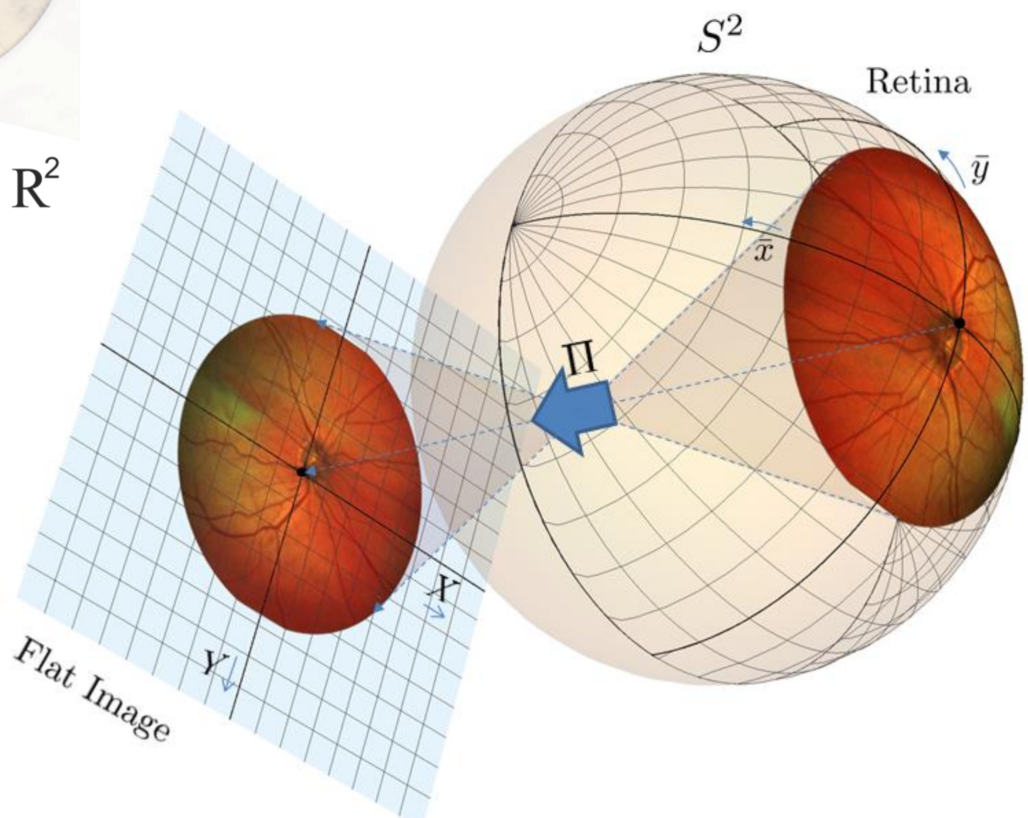
(with R. Duits, Y.L. Sachkov and I. Beschastnyi)

Sub-Riemannian Geodesics on SO(3)

Aim: data-driven SR geodesics on **SO(3)** for detection and analysis of vessel tree in spherical images of retina, to reduce distortion.



Spherical extension of cortical based model of perceptual completion on retinal sphere



Problem **Pcurve**(S^2) and **Pmec**(SO(3))

Given $\xi > 0, \quad \mathbf{n}_i \in S^2$

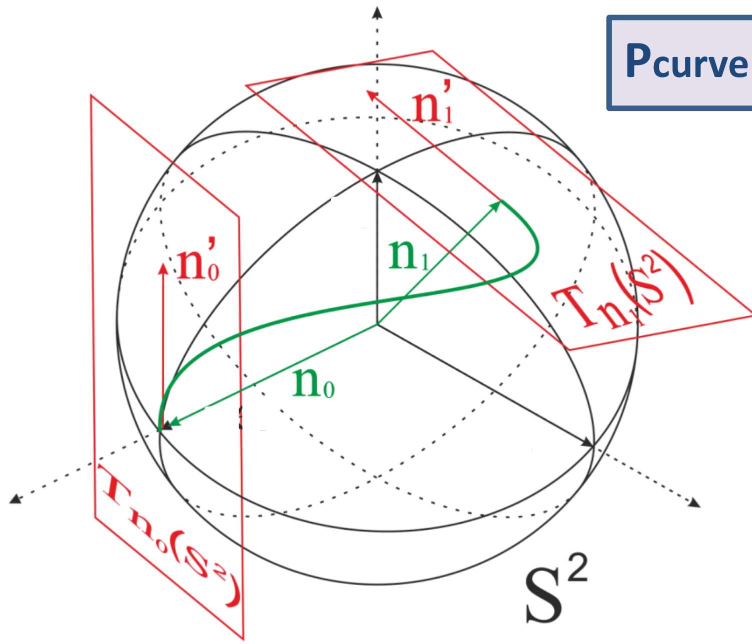
$\mathbf{n}_i' \in T_{\mathbf{n}_i} S^2$

Find a smooth curve $\gamma : [0, l] \to S^2$ s. t.:

$$\gamma(0) = \mathbf{n}_0, \quad \gamma(l) = \mathbf{n}_1,$$

$$\gamma'(0) = \mathbf{n}_0', \quad \gamma'(l) = \mathbf{n}_1',$$

$$E(\gamma(\cdot)) = \int_0^l \sqrt{\xi^2 + k_g^2(s)}\, \mathrm{d}s \to \min,$$

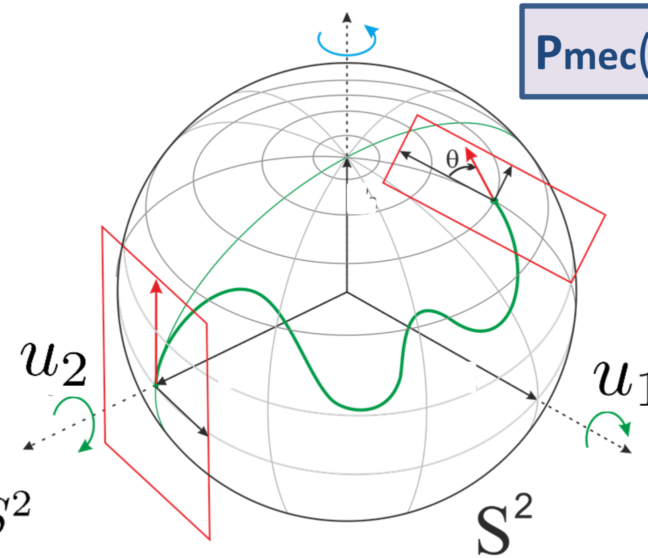


$$\dot{R} = -u_1 R A_2 + u_2 R A_1,$$

$$R(0) = \mathrm{Id}, \; R(t_1) = R_1,$$

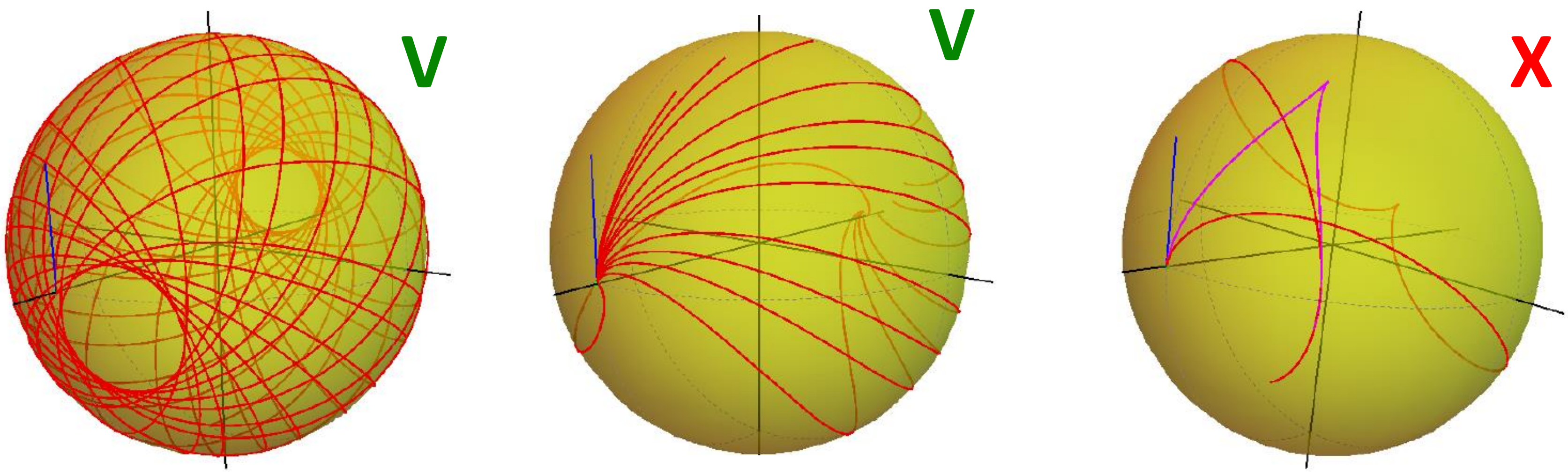
$$l(R(\cdot)) = \int_0^{t_1} \sqrt{\xi^2 u_1^2 + u_2^2}\, dt \to \min,$$

$$R \in \mathrm{SO}(3), \quad (u_1, u_2) \in \mathbb{R}^2$$



Spherical projection: $\mathrm{SO}(3) \ni R \mapsto Re_1 \in S^2$

SR geodesics in SO(3) with cuspless spherical projections



- Lift $P_{\text{curve}}(S^2)$ to sub-Riemannian problem on SO(3);
- Hamiltonian system of PMP;
- Classification by different dynamic of vertical part on elliptic ($0 < \xi < 1$), linear ($\xi = 1$) and hyperbolic ($\xi > 1$) cases;
- Explicit expressions for SR-geodesics in both SR-arclength and spherical arclength parameterization;
- Evaluation of first cusp time and description of reachable end conditions;
- Comparison cusp-surfaces and wavefronts w.r.t. SE(2)

Sub-Riemannian problem on SE(3) with cuspless spatial projection constraint

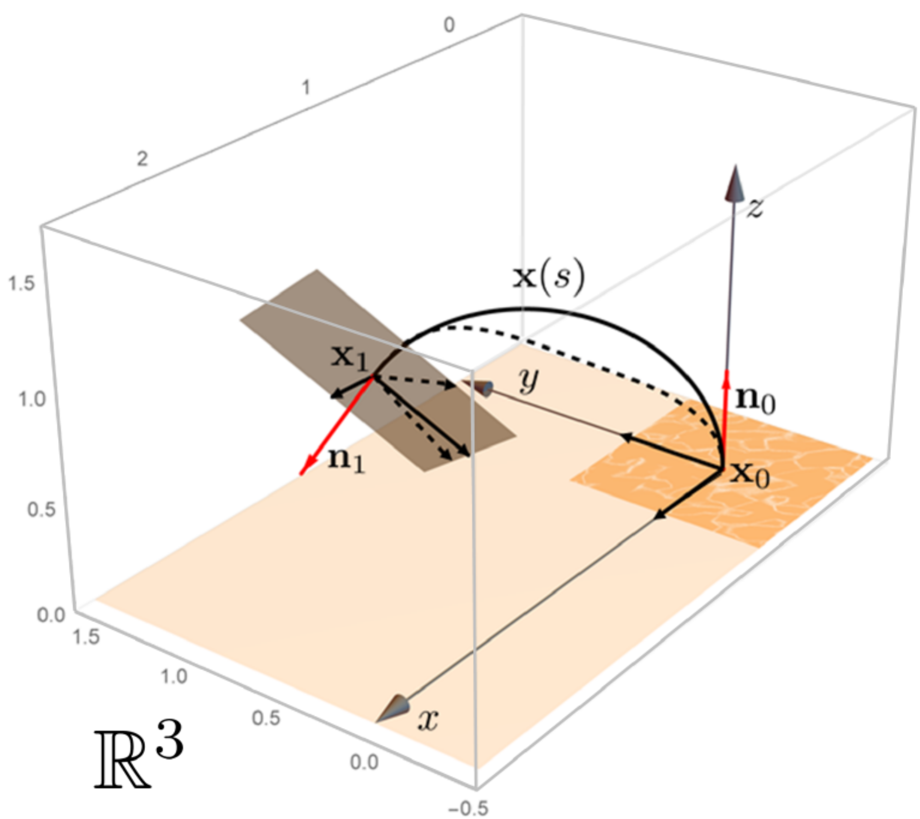
(with R. Duits, A. Ghosh and T.C.J. Dela Haije)

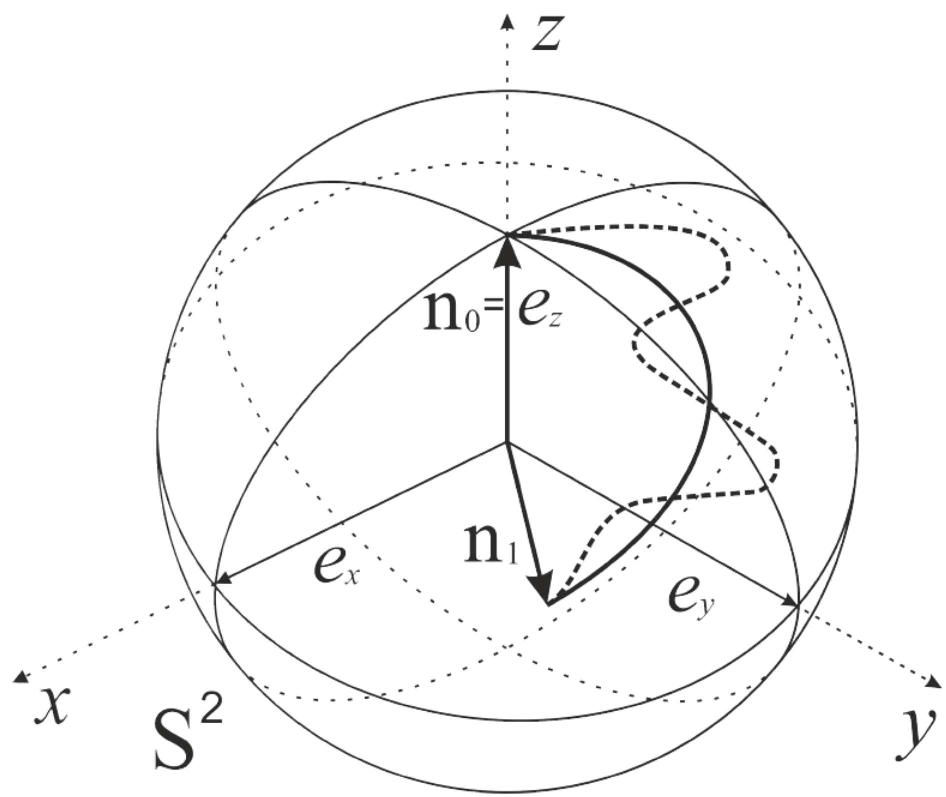
Problem **Pcurve($\mathbb{R}^3$)**: Shortest Path on $\mathbb{R}^3 \times S^2$

Given $\xi > 0, \quad \mathbf{x}_i \in \mathbb{R}^3, \ \mathbf{n}_i \in S^2, \ i \in \{0,1\}.$

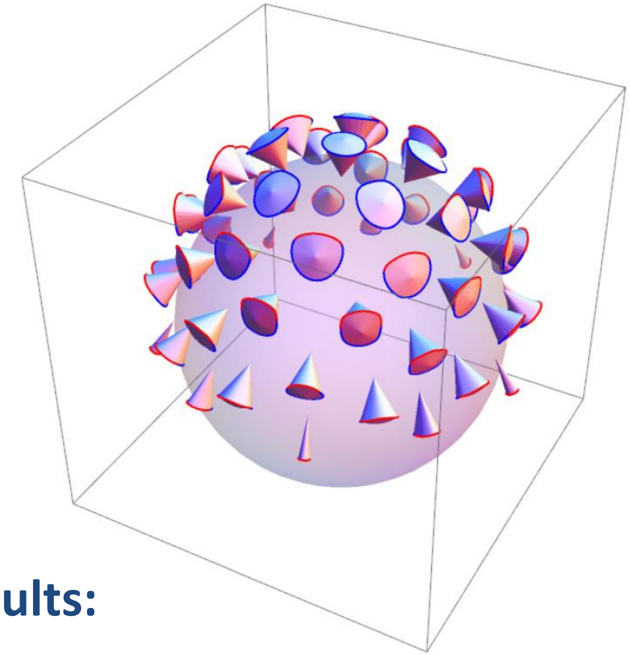
Find a smooth curve $\mathbf{x} \in C^\infty([0,L], \mathbb{R}^3)$ s.t. $\mathbf{x}(0) = \mathbf{x}_0, \ \mathbf{x}(L) = \mathbf{x}_1 \in \mathbb{R}^3,$ $\mathbf{x}'(0) = \mathbf{n}_0, \ \mathbf{x}'(L) = \mathbf{n}_1 \in \mathrm{S}^2,$

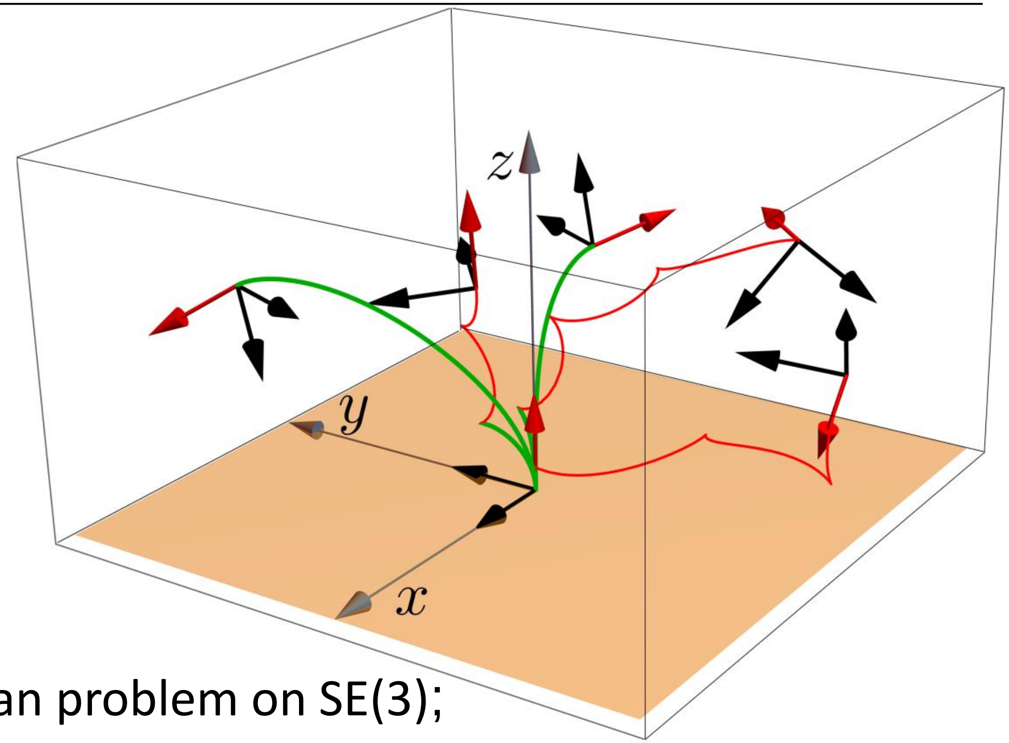
and $E(\mathbf{x}) := \int_0^L \sqrt{\xi^2 + \kappa^2(s)}\, \mathrm{d}s \to \min,$ where $\kappa(s) = \|\mathbf{x}''(s)\|.$





SR-geodesics on SE(3) with cuspless spatial projections





Results:

- Lift $P_{\mathrm{curve}}(\mathbb{R}^3 \times S^2)$ to sub-Riemannian problem on SE(3);
- Hamiltonian system of PMP;
- Liouville integrability of the Hamiltonian system;
- Explicit expressions for SR-geodesics in spatial arclength parameterization;
- Evaluation of first cusp time;
- Admissible boundary conditions reachable by cuspless geodesics;
- Geometrical properties: bounds on torsion, planarity conditions, symmetries;
- Numerical investigation of absence of conjugate points;
- Numerical solution to the boundary value problem.

Thank you for your attention!