Wardrop′s equilibrium and adjustment of OD matrices

Farhad Hatami

Universitat Autnoma de Barcelona

hatami@mat.uab.cat

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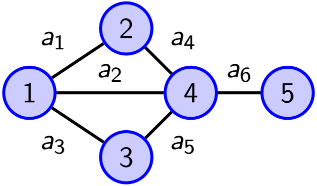
Principles of traffic modelling in static network

Graph shows a city (=network)
with 5 "centroids" (1,2,3,4,5),
6 "links" (a_1,a_2,a_3,a_4,a_5,a_6),
2 "Origin-Destination (OD) pairs" (1,5)
and 3 "paths" ($[a_1 a_4 a_6]$, $[a_1 a_2 a_6]$, $[a_3 a_5 a_6]$).



An OD matrix is a matrix whose entries g_{ij} represent how many cars go from origin i to destination j.

$$\begin{array}{c} \\ 1 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 5 \end{array} \\ \begin{bmatrix} 0 & 190 \\ 570 & 0 \end{bmatrix} \end{array}$$

Travel time on a link of the network
depends on flow $t(f)$ - cost function chosen by modeller.
Our problem is to determine which paths will be chosen by drivers and will be solved by "Wardrop's equilibrium".

Wardrop's principle

All the paths sharing the same origin and destination take the same equal travel cost (we are assuming cost as time).

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Braess's Paradox: 4000 vehicles go from START to END (N: total number of cars, t: corresponding time for each link).

Without dashed link
A, B: 2000 vehicles
time: 65

Dashed link with cost 0
START $\Rightarrow$ A $\Rightarrow$ B $\Rightarrow$ END: 4000 vehicles
time: 80

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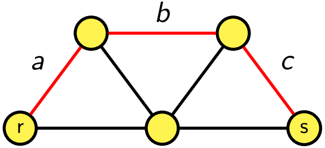
Given an OD pair (r, s):

- a is a link.
- $q_{r,s}$ = Total flow from r to s (number of cars go from r to s).
- $q = (q_{r,s})_{r,s}$ is the OD matrix.
- $f_{r,s}^{k}$ = flow from r to s through path k ($k = a \cup b \cup c$), is **UNKNOWN**.
- $f = (f_{r,s}^{k})_{r,s}$= Total flow for all pairs (r, s), from r to s through path k ($k = a \cup b \cup c$), is **UNKNOWN**.
- $c_{r,s}^{k} = \sum_a t_a(f_{r,s}^{k})\delta_{r,s}^{a,k}$ = travel time of path k.
- $v_a = \sum_{r,s,k} f_{r,s}^{k}\delta_{r,s}^{a,k}$ = flow through a link a.



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Given an OD matrix q, our goal is to minimize the total time (=cost)

$$\min_{f=(f_{r,s}^k)_{r,s}} \sum_a \int_0^{v_a(f)} t_a(w)\mathrm{d}w$$

with the following constraints

$$\sum_k f_{r,s}^k = q_{r,s} \qquad \forall r, s$$

$$f_{r,s}^k \geq 0 \qquad \forall r, s, k$$

Remark

Given $q = q(r,s)_{r,s}$ we obtain a unique minimizer $f = f(q)$.

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Assignment problem

Solution of previous minimization problem leads to equilibrium paths. Once we have found path flows ($f = (f^k_{r,s})_{r,s}$), then we can find link flows ($v^*_a(q) = v^*_a(q(f))$). This particular link flow $v^*_a(q)$ is called the "**assigned flow**".

OD estimation using statistical methods

OD estimation is then solved using gradient descent with the problem

$$\min_{q=(q_{r,s})_{r,s}} \sum_a (v_a - v_a^*(q))^2$$

where v_a is the real data (or observed flow) and $v_a^*(q)$ is the assigned flow.

There are 2 steps to solve the problem:

1. Calculating gradient,
2. Evaluating the new point = Solving the assignment problem.

OD estimation using statistical methods

Consider 10^6 OD pairs and 10^2 observations. Then there are many solutions (a system with number of variables hugely more that data).
In order to tackle this issue, sparsity of assigned matrix should be increased.
So new constraints are as follows:

1. Not far from the initial matrix,
2. bounded values,
3. non-negative q (step small enough),
4. zeros are not modified (step proportional to the value) (there is no notion of time).

OD estimation using statistical methods

These new constraints lead to the following problem

$$\min_q \sum_a (v_a - v_a^*(q))^2 + \sum_{r,s} (q_{r,s} - q_{r,s}^0)^2$$

where $q^0 = (q_{r,s}^0)_{r,s}$ is the initial OD matrix.
Using gradient descent method and iteration, the best matrix q which is fit to the initial OD matrix q^0 could be found.
We are investigating which one of the following minimization problem is better to fit the real flow to the assigned flow and simultaneously to fit the adjusted matrix to the initial OD matrix:

A

$$\min_q \sum_a (v_a - v_a^*(q))^2 + \lambda \sum_{r,s} |q_{r,s} - q_{r,s}^0|$$

B

$$\min_q \sum_a (v_a - v_a^*(q))^2 + \lambda \sum_{r,s} |q_{r,s}|$$

OD estimation using statistical methods

A

$$\min_q \sum_a (v_a - v_a^*(q))^2 + \lambda \sum_{r,s} |q_{r,s} - q_{r,s}^0|$$

B

$$\min_q \sum_a (v_a - v_a^*(q))^2 + \lambda \sum_{r,s} |q_{r,s}|$$

Here there is a penalization error in both A and B, which is multiplied by a penalization constant. This constant could be estimated directly by well-know "cross-validation" method in statistics.

OD estimation using statistical methods

Adding this penalization term has some benefits and drawbacks:

A

$$\min_q \sum_a (v_a - v_a^*(q))^2 + \lambda \sum_{r,s} |q_{r,s} - q_{r,s}^0|$$

Using A, obtained matrix will be preserved close to the initial OD matrix (q^0), provided λ is chosen large enough.
Initial OD matrix is valuable because it comes from our survey, so it is important to find matrix which is not so far away from the initial one.

OD estimation using statistical methods

B

$$\min_{q} \sum_{a} (v_a - v_a^*(q))^2 + \lambda \sum_{r,s} |q_{r,s}|$$

Using B, our problem will be simplified because many zeroes will be produced ($q_{r,s} = 0$) and sparsity of matrix will be increased (in the process of calculation, number of real data will be close to the number of variables).

OD estimation using statistical methods

A

$$\min_q \sum_a (v_a - v_a^*(q))^2 + \lambda \sum_{r,s} |q_{r,s} - q_{r,s}^0|$$

B

$$\min_q \sum_a (v_a - v_a^*(q))^2 + \lambda \sum_{r,s} |q_{r,s}|$$

Using A and B have some drawbacks. Especially, ℓ^1 norm is not differentiable at zero, so it can be replaced with smooth function, using convolution with Gaussian function (that we are working on).

OD estimation using statistical methods

At present, we are working on minimizing relative errors (non-smooth problem)

$$\min_{q} \sum_{a} \frac{1}{v_a} |v_a - v_a^*(q)|$$

Thank you for your attention