Stable area-stationary surfaces in sub-Riemannian geometry

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In this talk I want to present

1. the regularity of critical points of the sub-Riemannian (SR) area functional in 3-dimensional contact SR manifolds
2. the second order minima of the SR area functional in $\mathbb{H}^1$
3. the area-minimizing property of stable minimal graph in $\mathbb{H}^n$

These results are contained in

- M. Galli, M. Ritoré, *Area-stationary and stable surfaces of class C^1 in the sub-Riemannian Heisenberg group $\mathbb{H}^1$*, Adv. Math., 2015
- M. Galli, M. Ritoré, *Regularity of C^1 surfaces with prescribed mean curvature in three-dimensional contact sub-Riemannian manifolds*, Calc. Var. and PDE, 2015
- M. Galli, *Regularity of Lipschitz graphs with prescribed mean curvature in three-dimensional contact sub-Riemannian manifolds*, preprint
- G. Citti, M. Galli, *The area minimizing property of stable minimal graphs in $\mathbb{H}^n$*, preprint

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1. Contact sub-Riemannian manifold $(M, \mathcal{H}, g_{\mathcal{H}})$

- M: $(2n+1)$-dimensional manifold
- ω: contact 1-form
- $\mathcal{H} = \mathrm{Ker}(\omega)$: horizontal distribution
- J: holomorphic involution on $\mathcal{H}$
- $g_{\mathcal{H}}$: positive definite metric on $\mathcal{H}$,
- T: Reeb vector field, defined by $\omega(T) = 1$ and $\mathcal{L}_T\omega = 0$

$g_{\mathcal{H}}$ is extended to a Riemannian metric g by making T unitary and orthogonal to $\mathcal{H}$

1.1 The geometry of C^1 surfaces in M

$\Sigma \subset M$ immersed surface of class C^1. Define

$$\Sigma_0 := \{p \in \Sigma; T_p\Sigma = \mathcal{H}_p\} \qquad \text{(singular set)},$$

$$N : \qquad \text{(Riemannian unit normal)},$$

$$N_H := N - g(N,T)T \qquad \text{(horizontal projection of } N\text{)},$$

$$\nu_H := \frac{N_H}{|N_H|} \qquad \text{(horizontal unit normal in } \Sigma - \Sigma_0\text{)},$$

$$Z := J(\nu_H) \in T(\Sigma - \Sigma_0) \qquad \text{(characteristic field)},$$

1.2 The sub-Riemannian area and the mean curvature

1. Area: N Riemannian normal, N_H horizontal projection of N, $d\Sigma$ Riemannian measure of Σ in (M, g)

$$A(\Sigma) := \int_\Sigma |N_H| \, d\Sigma.$$

2. Mean curvature: $\Sigma = \partial E \subset \Omega$ Lipschitz, Σ has mean curvature f if it is a critical point of

$$A(\Sigma \cap B) - \int_{E \cap B} f \, dv,$$

for any bounded open set $B \subset \Omega$, dv Riemannian volume element on M.

3. If $\Sigma \in C^2_h$ the mean curvature H of Σ coincides with

$$H := -\operatorname{div}^h_\Sigma(\nu_H).$$

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1.3 Previous results

Few regularity results:

- Cheng-Hwang-Yang '09: the regular part $\Sigma - \Sigma_0$ of a minimal graph of class C^1 in $\mathbb{H}^1$, with $g(N, T) \neq 0$, is foliated by horizontal straight lines
- Capogna-Citti-Manfredini '09, Barbieri-Citti '11: minimal regular Lipschitz graphs that are limits of Riemannian minimal graphs in M, $n = 1$, are foliated by horizontal smooth curves
- Capogna-Citti-Manfredini, '09: minimal regular Lipschitz graphs that are limits of Riemannian minimal graphs in $\mathbb{H}^n$, $n > 1$, are smooth

Many examples of complete area-minimizing surfaces with low regularity:

- (Cheng-Hwang-Yang '07, Barone Adesi-Serra Cassano-Vittone '07, Ritoré '09) Plenty of $C^{0,1}$ minimizers, depending on an arbitrary monotone function $f : \mathbb{R} \to \mathbb{R}$

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2. Regularity of critical points

Theorem

Let $\Omega \subset M$ be a domain, $n = 1$, and let $E \subset \Omega$ be a set of prescribed mean curvature $f \in C^k(\Omega)$ with Euclidean Lipschitz boundary $\partial E = \Sigma$. If Σ is locally a Lipschitz regular graph, then the characteristic curves are of class C^{k+2}.

Idea of the proof

We localize the first variation of the area along a characteristic curve

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3. The Stability operator

A minimal hypersurface $\Sigma \subset M$, with $\Sigma_0 = \emptyset$, is stable (resp. strictly stable) if and only if

$$\mathcal{I}(v) = -\int_{\Sigma} v\,\mathcal{L}(v)\,(|N_H|d\Sigma) \geq (\text{resp. } >)0$$

for any non-trivial $v \in C_h^2(\Sigma)$, where $\mathcal{L}$ is the second order linear sub-elliptic operator

$$\mathcal{L}(v) := \frac{d}{dt}\bigg|_{t=0} H_t = \Delta_\Sigma^h(v) + 2\frac{\langle N,T\rangle}{|N_H|}\langle \nabla_\Sigma^h(v), J(\nu_h)\rangle + qv.$$

In the Heisenberg group $\mathbb{H}^n$

$$q = |\sigma|^2 + 4Z\left(\frac{\langle N,T\rangle}{|N_H|}\right) + (2n+2)\left(\frac{\langle N,T\rangle}{|N_H|}\right)^2.$$

3.1 A technical regularity result in $\mathbb{H}^n$

The key point to write the operator $\mathcal{L}$ is to improve the "a priori" regularity of some geometric quantities along the characteristic direction

Proposition (G-Ritoré)

The function $v = g(N, T)/|N_H|$ is smooth when restricted to characteristic curves in a C^1 minimal surface $\Sigma \subset \mathbb{H}^1$.

Proposition (Citti-G)

A minimal C_h^2 graph $\Sigma \subset \mathbb{H}^n$, $n > 1$, is smooth.

3.2 Stable area-stationary surfaces and the Bernstein problem

Theorem

Let $\Sigma \subset \mathbb{H}^1$ be a complete oriented stable minimal surface of class C^1 without singular points. Then Σ is a vertical plane.

A consequence is the following Bernstein Theorem for regular graphs of C^1 functions

Theorem

Let $\Sigma \subset \mathbb{H}^1$ be a complete locally area-minimizing regular graph of a C^1 function. Then Σ is a vertical plane.

Remark

These results are well known for C^2 surfaces in $\mathbb{H}^1$ (Barone Adesi-Serra Cassano-Vittone '07, Danielli-Garofalo-Nhieu-Pauls '07, '10, Hurtado-Ritoré-Rosales '10).

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4. Area-minimizing property of Σ

Theorem

Let Σ be a $C_h^{2,\alpha}$ minimal graph in $\mathbb{H}^n$. We assume that $\Sigma_0 = \emptyset$ and Σ is strictly stable in a domain $\Omega \subset \Sigma$.
Then Σ is locally area-minimizing (there exists a tubular neighborhood U of Ω such that for any C_h^1 graph $S \subset U$, $\partial\Omega = \partial S$, we have $\mathrm{A}(\Omega) \leq \mathrm{A}(S)$ or $\Omega = S$).

Idea of the proof

Using Schauder local estimates of $\mathcal{L}$, we construct a calibration of Σ in U that implies the area-minimizing property

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Thanks for your attention!!!