A neurogeometrical model for image completion and visual illusion

Benedetta Franceschiello
Advisors: A. Sarti, G. Citti

CAMS (Unité Mixte CNRS-EHESS), University of Bologna

Mid-term review meeting of MAnET project
Helsinki, Dec 9th, 2015

Objectives

Mathematical models for low-level vision to perform:

- (i) Amodal completion (inpainting), enhancement;
- (ii) Visual perception of geometrical optical illusion.

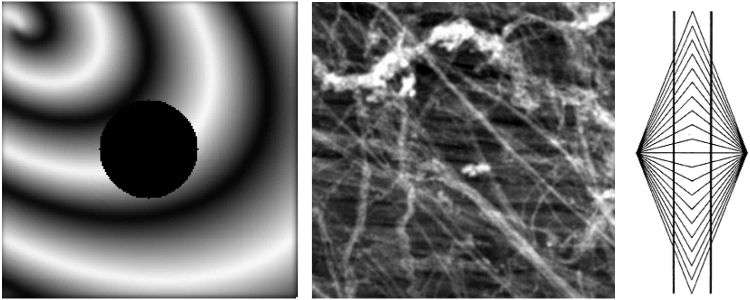


Figure: Inpainting, enhancement and a GOI

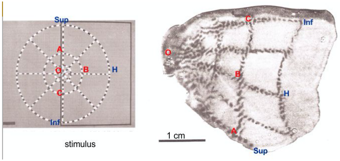
Functional architecture of the primary visual cortex

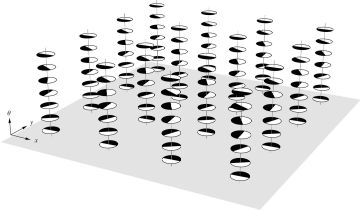
Primary visual cortex (V1):
Elaborates information from the retina

- Retinotopic Structure;
- Hypercolumnar Structure

Connectivity:

- Intra-cortical
- Long range connection



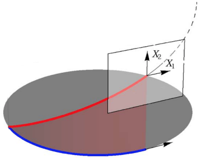


Cortical based Model[1]

V1 as rototranslation group SE(2)=$\mathbb{R}^2 \times S^1$:

- $(x, y) \in \mathbb{R}^2$ represents a position on the retina;
- If $\widetilde{\gamma}(t) = (x(t), y(t))$ is a visual stimulus on the retina, the hypercolumn over $(x(t), y(t))$ selects the tangent direction θ
- The tangent vectors to any lifted curve $\gamma(t) = (x(t), y(t), \theta(t))$ are a linear combination of:

$$X_1 = \begin{pmatrix} cos\theta \\ sen\theta \\ 0 \end{pmatrix} X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$\nexists$ "lifted curves " with tangent direction along $X_3 = [X_1, X_2]$

[1] G. Citti, A. Sarti, *J. Math. Imaging Vision 24* (2006)

- The connectivity model is given by a 2-dimensional subspace of the tangent space of $SE(2)$: X_1 e $X_2 \in HM \subset T(SE(2))$
- Then we define a metric on HM:

$$\|\alpha_1 X_1 + \alpha_2 X_2\|_g = \sqrt{\alpha_1^2 + \alpha_2^2}$$

- Its Riemannian completion is:

$$\|\alpha_1 X_1 + \alpha_2 X_2 + \varepsilon\alpha_3 X_3\|_{g_\epsilon} = \sqrt{\alpha_1^2 + \alpha_2^2 + \varepsilon^2\alpha_3^2}$$

obtaining the previous expression for $\varepsilon \to 0$.

$$(g^{ij}) = \begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) & 0 \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Mean curvature flow

- Reconstruction of perceptual phenomena and modeling of the visual signal through mean curvature flow.
- Sub-Riemannian mean curvature flow

$$\begin{cases} u_t = \sum_{i,j=1}^{2} \left(\delta_{i,j} - \frac{X_i^0 u X_j^0 u}{|\nabla_0 u|^2} \right) X_i^0 X_j^0 u \\ u(\cdot, 0) = u_0 \end{cases}$$

- S. Osher and J.A. Sethian[2]; L.C. Evans and J. Spruck[3].
- Theorem: *There exist viscosity solutions uniformly Lipshitz-continuous to the mean curvature flow in SE(2)*[4].

[2] *J.Computational Phys. 79*, (1988);
[3] *J. Differential Geom. 33* (1991)
[4] Citti, F., Sanguinetti, Sarti, *Accepted by SIAM J. Imaging Sciences* (2015);

Proof

$$\begin{cases} u_t = \sum_{i,j=1}^{3} \left(\delta_{i,j} - \frac{X_i^\epsilon u X_j^\epsilon u}{|\nabla_\epsilon u|^2 + \tau} + \sigma \delta_{i,j} \right) X_i^\epsilon X_j^\epsilon u \\ u(\cdot, 0) = u_0 \end{cases}$$

We look for solutions $u^{\epsilon,\tau,\sigma}$ and uniform estimates for the gradient[5]

$$\|u^{\epsilon,\tau,\sigma}(\cdot, t)\|_{\mathcal{L}^\infty(\mathbb{R}^2 \times S^1)} \leq \|u_0\|_{\mathcal{L}^\infty(\mathbb{R}^2 \times S^1)}$$

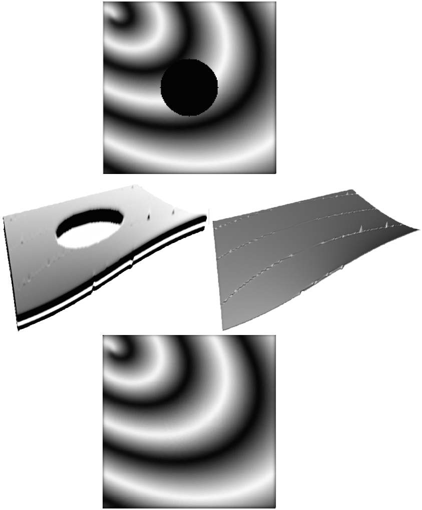
$$\|\nabla_E u^{\epsilon,\tau,\sigma}(\cdot, t)\|_{\mathcal{L}^\infty(\mathbb{R}^2 \times S^1)} \leq \|\nabla_E u_0\|_{\mathcal{L}^\infty(\mathbb{R}^2 \times S^1)}$$

Then $\epsilon, \tau, \sigma \to 0$ to recover a vanishing viscosity solution in the space of Lipshitz functions to the initial problem.

[5] Capogna, Citti, *Communications in Partial Diff. Equations V. 34* (2009); Ladyženskaja, Solonnikow, Ural'ceva, *American Mathematical Soc.*(1988)

Image Processing

- The missing part is a minimal surface.
- We lift and we let the image evolve through mean curvature flow
- the gray-levels are lifted to a function v defined on the surface.
- Laplace-Beltrami of v is used to complete the color;



Results[6]

Figure: From left to right: The original image, The image processed in [6], Inpainting performed with our algorithm.

[6] Comparison made with: Boscain, Chertovskih, Gauthier, Remizov, *SIAM J. Imaging Sciences*; (2014)

Results[7]

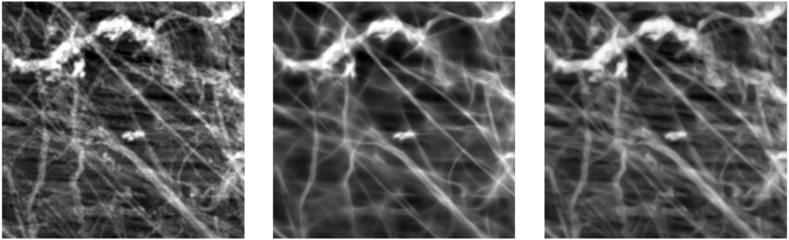
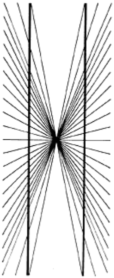


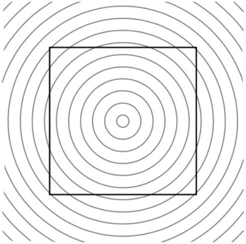
Figure: From left to right: the original image, the image processed through CED-OS, Enhancement with our algorithm.

[7] Comparison made with: Duits, Franken, *Quarterly on Applied Mathematics 68(2)*; (2010)

Geometrical-optical illusions and literature

Geometrical–optical illusions are situations in which there is an awareness of a mismatch of geometrical properties between an item in object space and its associated percept. (Oppel [8])





[8] Westheimer, *Vision Research 48*; (2008)

History of the problem

Ehm, Wackermann[9]:

- Model of Hering-type illusions as geodesics
- Regression to right angles
- Background without crossing lines

Yamazaki, Yamanoi[10]:

- Use of deformations for Delbouf illusion

Objectives:

- To overcome the limitations
- To take into account the cortical behaviour

[9] *J. of Mathematical Psychology*; (2013)
[10] *Bchaviormetrika v.26*; (1999)

The idea under the model

The deformation is a map: $\phi : (\mathbb{R}^2, (p_{ij})_{i,j=1,2}) \rightarrow (\mathbb{R}^2, Id_{\mathbb{R}^2})$
We would like to:

- recover it as a displacement field $\{\bar{u}(x, y)\}_{(x,y)\in\mathbb{R}^2}$
- study how the metric $(p_{ij})_{i,j=1,2}$ changes

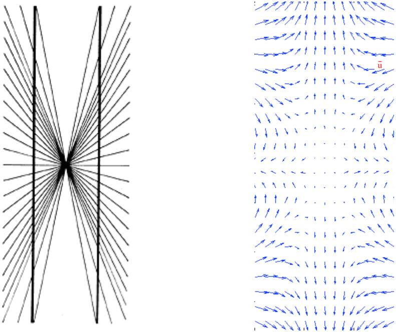


Figure: The illusion is interpreted as an elastic deformation (strain)

What is p_{ij}?

The strain theory on $\mathbb{R}^2$ is induced by the cortical structure:

$$p = \int_0^\pi \exp^{-\frac{((\sin(\theta-\bar{\theta}))^2)}{2\sigma}} \cdot \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} d\theta$$

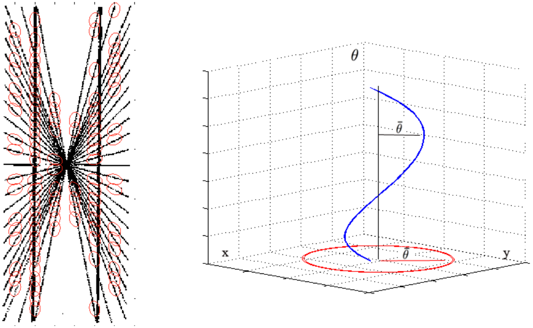


Figure: The maximum activity is registered at $\bar{\theta}$

From strain to displacement

Then from the infinitesimal strain theory we have:

- $p = (\nabla\phi)^T \cdot (\nabla\phi)$ where $(\nabla\phi)$ is the deformation gradient
- From $\phi(x,y) = \bar{u}(x,y) + Id$ we obtain

$$(p - Id)(x,y) = \nabla\bar{u}(x,y) + (\nabla\bar{u}(x,y))^T$$

- Differentiating and substituting:

$$\begin{cases} \Delta u &= -\partial_x(p_{22}) + \partial_x(p_{11}) + 2\partial_y(p_{12}) := \alpha_1 \\ \Delta v &= -\partial_y(p_{22}) + \partial_y(p_{11}) + 2\partial_x(p_{12}) := \alpha_2 \end{cases}$$

- Solving numerically the Poisson problems we recover the displacement field $\{\bar{u}(x,y)\}_{(x,y)\in\mathbb{R}^2}$.

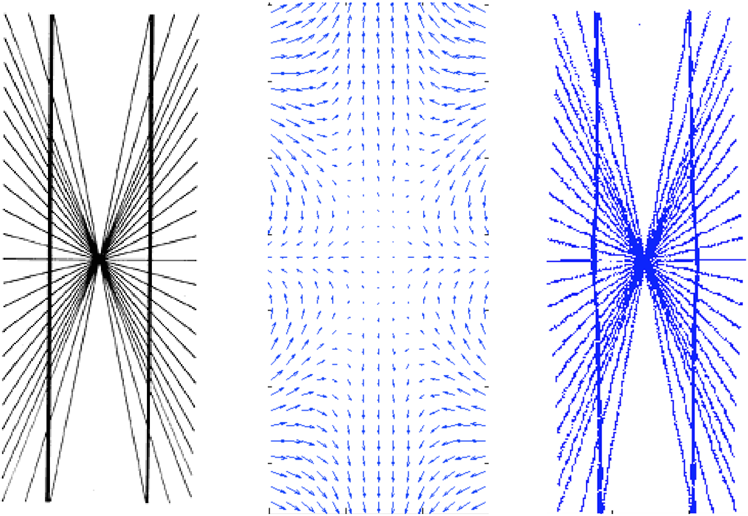


Figure: Perceived deformation for the Hering illusion.

Work in Progress

- Interpretation of deformed lines as geodesic in the $\mathbb{R}^2 \times S^1$
- Completion model and strain model applied to the Poggendorff illusion: