Currents in the Heisenberg Group

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Contents

Plateau Problem

Definition (Plateau Problem)

Given a boundary with some kind of regularity, the Plateau Problem consists in finding the minimal surface that fits that boundary.

How is it related with currents? We'll see an example soon.

Riemannian Currents

Definition (Riemannian Currents)

Let M be a Riemannian manifold and $\mathcal{D}^k(M)$ the space of smooth $k-$forms.
The dual space is called the space of $k-$dimensional currents.

Observation

Any oriented $k-$dim rectifiable set S can be seen as a current:

$$[\![S]\!] : \varphi \in \mathcal{D}^k(M) \longrightarrow \int_S \langle \overrightarrow{S}(x), \varphi \rangle \mu(x) d\mathcal{H}^k \in \mathbb{R}$$

Definition (Rectifiable Currents)

Requiring also $\mathbf{M}([\![S]\!]) < \infty$, we call such a $[\![S]\!]$ a rectifiable current and write $[\![S]\!] \in \mathcal{R}_k(M)$.

Riemannian Currents

Definition (Currents mod p)

Consider two rectifiable currents $T_1, T_2 \in \mathcal{R}_k(M)$. We say that T_1 and T_2 are congruent modulo p, and write $T_1 \equiv T_2 \bmod p$, *if and only if*

$$\mathcal{F}^p(T_1 - T_2) = 0$$

where

$$\mathcal{F}^p(T) = \inf\{\mathbf{M}(R) + \mathbf{M}(S)\}$$

with $T = R + \partial S + pQ$, $R \in \mathcal{R}_k(M)$, $S \in \mathcal{R}_{k+1}(M)$ and $Q \in \mathcal{R}_k(M)$.

2015 Casteras, Holopainen, Ripoll

And here is our example:

Theorem (2015 Casteras, Holopainen, Ripoll)

Let M^n, $n \geqslant 3$, be a Cartan-Hadamard manifold satisfying the SC condition and let $\Gamma \subset \partial_\infty M^n$ be a (topologically) embedded closed $(k-1)-$dimensional submanifold, with $2 \leqslant k \leqslant n-1$. Then there exists a complete, absolutely area minimizing, locally rectifiable $k-$current Σ modulo 2 in M^n asymptotic to Γ at infinity, *i.e.* $\partial_\infty spt\Sigma = \Gamma$.

Notation

Notation

With the usual notation of the Heisenberg group $\mathbb{H}^n$, we denote
$\mathfrak{h} := \langle X_1, \dots, X_n, Y_1, \dots, Y_n, T \rangle \equiv \langle W_1, \dots, W_{2n+1} \rangle$
and we call set of $k-$vectors, $1 \leqslant k \leqslant 2n+1$,

$$\bigwedge\nolimits_k \mathfrak{h} := \langle W_{i_1} \wedge \dots \wedge W_{i_k} \rangle_{1 \leqslant i_1 < \dots < i_k \leqslant 2n+1}.$$

Indicating $\bigwedge^1$ as the dual space, we say that

$$\bigwedge\nolimits^1 \mathfrak{h} = \langle \theta_1, \dots, \theta_{2n+1} \rangle$$

and then define the set of $k-$covectors as

$$\bigwedge\nolimits^k \mathfrak{h} := \langle \theta_{i_1} \wedge \dots \wedge \theta_{i_k} \rangle_{1 \leqslant i_1 < \dots < i_k \leqslant 2n+1}.$$

Notation

Definition (Integrable $k-$vectors, 2007 Franchi at al., 2.5)

Let $1 \leqslant k \leqslant n$. Then

$$_H\bigwedge_0 := \mathbb{R}$$

$$_H\bigwedge_k := span\left\{v \in \bigwedge_k \mathfrak{h}_1 \mid v \text{ simple and integrable}\right\}$$

$$_H\bigwedge_{2n+1-k} := *_H\bigwedge_k$$

Definition (Integrable $k-$covectors)

$$_H\bigwedge^k := \bigwedge^1\left(_H\bigwedge_k\right)$$

Notation

Definition (from 1969 Federer, 1.7.8)

Let $1 \leqslant k \leqslant 2n$. We define the linear isomorphisms

$$* : \bigwedge_k \mathfrak{h} \longrightarrow \bigwedge_{2n+1-k} \mathfrak{h}$$
$$v \longmapsto *v$$

where $v = \sum_I v_I W_I$ and $*v = \sum_I v_I (*W_I)$ and

$$*W_I := (-1)^{\sigma(I)} W_{I*}$$

with $I^* = \{1, \dots, 2n+1\} \smallsetminus I$, $\sigma(I) \in \mathbb{N}$.

Sub–Riemannian Currents

Definition

Let $U \subset \mathbb{H}^n$ open. We define $\mathcal{D}^k_{\mathbb{H}}(U)$, the set of *Heisenberg k−differentiable forms*, as the space of compactly supported smooth sections of ${}_H \bigwedge^k$.

Definition

We call *Heisenberg k−current*, $1 \leqslant k \leqslant 2n+1$, any continuous linear functional on $\mathcal{D}^k_{\mathbb{H}}(U)$.

In a more general setting...

Definition (Currents in a Metric Space)

Let E be a metric space and $k \geqslant 0$. A *$k-$dimensional metric current* is a $k-$dimensional metric functional with finite mass s.t.

- $T(f, \pi_1, \dots, \pi_k)$ is multilinear with $f \in Lip_b(E)$ and $\pi_1, \dots, \pi_k \in Lip(E)$
- $\lim_{i \to \infty} T(f, \pi_1^i, \dots, \pi_k^i) = T(f, \pi_1, \dots, \pi_k)$, with $\pi_j^i \to \pi_j$ pointwise in E and $Lip(\pi_j^i) \leqslant c < \infty$.
- $T(f, \pi_1, \dots, \pi_k) = 0$ if a π_i is constant on a neighbourhood of $\{f \neq 0\}$.

As already stated, our main goal is to study how rectifiable currents (eventually *mod p*) can help solving some kind of Plateau Problem (Casteras&Holopainen&Ripoll).

More in detail, our current intermediate steps are

1. A deeper understanding of the definition of currents (Franchi et al.)
2. The study of Sub–Riemannian currents *mod p* (Morgan, Ambrosio&Kirchheim, Ambrosio&Wenger but also Franchi et al.)
3. Analysis of some properties of currents in sub-Riemannian setting (Franchi et al.)

Thank you!

Kiitos paljon!