Surface reconstruction via mean curvature flow

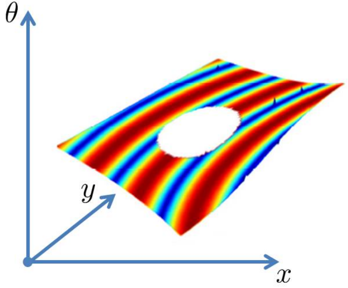
Emre Baspinar

supervised by prof. dr. Giovanna Citti

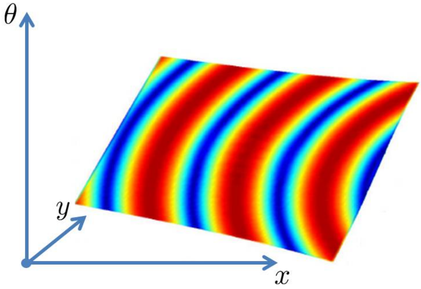
Department of Mathematics, University of Bologna

December 6, 2015

Physical motivation



Corrupted surface



Reconstructed surface

Outline

Part I

- Preliminaries
  - Sub-Riemannian geometry
  - Vanishing viscosity
- Literature on uniqueness
- Uniqueness in sub-Riemannian mean curvature flow

Part II

- Bence-Merriman-Osher algorithm
- Citti-Sarti diffusion driven motion
- A new diffusion driven motion
  - in Euclidean setting
  - in the sub-Riemannian setting
- Summary and future work

PART I

Uniqueness in the sub-Riemannian mean curvature flow

Preliminaries: SE(2) sub-Riemannian geometry

- Elements: $(x, y, \theta) \in \mathrm{SE}(2)$
- Horizontal plane: $\mathrm{span}\{X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y,\ X_2 = \partial_\theta\}$
- For $u : \mathrm{SE}(2) \to \mathbb{R}$
  - horizontal gradient: $\nabla_h u = (X_1 u, X_2 u)$
  - horizontal divergence: $\mathrm{div}_h\, \nu = X_1\nu_1 + X_2\nu_2$
  - horizontal unit normal: $\nu_h = \frac{\nabla_h u}{|\nabla_h u|} = \frac{(X_1 u,\ X_2 u)}{\sqrt{(X_1 u)^2 + (X_2 u)^2}}$
  - horizontal Laplacian: $\Delta_h u = X_1^2 u + X_2^2 u$
  - horizontal mean curvature: $K_h = \mathrm{div}_h(\nu_h) = \mathrm{div}_h \left(\frac{\nabla_h u}{|\nabla_h u|} \right)$

Preliminaries: SE(2) sub-Riemannian geometry

$$X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$

- Elements: $(x, y, \theta) \in \mathrm{SE}(2)$
- Horizontal plane: $\mathrm{span}\{X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y,\ X_2 = \partial_\theta\}$
- For $u : \mathrm{SE}(2) \to \mathbb{R}$
  - full gradient: $\nabla u = (X_1 u, X_2 u, X_3 u)$
  - full divergence: $\mathrm{div}\, \nu = X_1\nu_1 + X_2\nu_2 + X_3\nu_3$
  - full unit normal: $\nu = \frac{\nabla u}{|\nabla u|} = \frac{(X_1 u,\ X_2 u,\ X_3 u)}{\sqrt{(X_1 u)^2 + (X_2 u)^2 + (X_3 u)^2}}$
  - full Laplacian: $\Delta u = X_1^2 u + X_2^2 u + X_3^2 u$
  - full mean curvature: $K = \mathrm{div}(\nu) = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$

Degenerate!

Preliminaries: SE(2) sub-Riemannian geometry

$$X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$

- Elements: $(x, y, \theta) \in SE(2)$
- Horizontal plane: $\text{span}\{X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y,\ X_2 = \partial_\theta\}$
- For $u : SE(2) \to \mathbb{R}$
  - full gradient: $\nabla u = (X_1 u, X_2 u, X_3 u)$
  - full divergence: $\text{div}\,\nu = X_1\nu_1 + X_2\nu_2 + X_3\nu_3$
  - full unit normal: $\nu = \frac{\nabla u}{|\nabla u|} = \frac{(X_1 u,\ X_2 u,\ X_3 u)}{\sqrt{(X_1 u)^2 + (X_2 u)^2 + (X_3 u)^2}}$
  - full Laplacian: $\Delta u = X_1^2 u + X_2^2 u + X_3^2 u$
  - full mean curvature: $K = \text{div}(\nu) = \text{div}\left(\frac{\nabla u}{|\nabla u|}\right)$

Degenerate!

- Non-commutative Lie algebra:

$$[X_1,\ X_2] = -X_3 = \sin(\theta)\partial_x - \cos(\theta)\partial_y$$

Challenging but satisfies Hörmander condition!

Preliminaries: Sub-Riemannian mean curvature flow

$$\begin{cases} u_t = \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i u X_j u}{|\nabla_h u|^2} \right) X_{ij} u & \text{in } SE(2) \times (0, \infty) \\ u = u_0 & \text{on } SE(2) \times \{0\} \end{cases}$$

Characteristic points: $|\nabla_h u| = \sqrt{(X_1 u)^2 + (X_2 u)^2} = 0$

- Global description

BUT...

- Not defined when $\nabla_h u = 0$!
- Requires regularization

Preliminaries: Vanishing viscosity

Regularized equation

$$\begin{cases} u_t^\epsilon = \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i u^\epsilon X_j u^\epsilon}{\epsilon^2 + |\nabla_h u^\epsilon|^2} \right) X_{ij} u^\epsilon \\ u^\epsilon(.,0) = u_0(.) \end{cases}$$

No characteristic points!

Degenerate equation

$$\begin{cases} u_t = \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i u X_j u}{|\nabla_h u|^2} \right) X_{ij} u \\ u(.,0) = u_0(.) \end{cases}$$

Not defined when $\nabla_h u = 0$!

vanishing viscosity $\lim_{\epsilon \to 0} u^\epsilon$	=	viscosity u

Literature on uniqueness

- Euclidean, Evans-Spruck and Chen-Giga-Goto
- Euclidean, Deckelnick
- Heisenberg group, existence of graph, Capogna-Citti
- Heisenberg group, axisymmetricity, Ferrari-Liu-Manfredi

Problematic with characteristic points!

What about general setting?

Uniqueness in vanishing viscosity sense

1. $$\sup_{\xi \in \mathrm{SE}(2), 0 \leq t \leq T} \left| (u^{\epsilon_1} - u^{\epsilon_2})(\xi, t) \right| \qquad \text{attainable?}$$

2. Argue by contradiction:

   For all $M \geq 0$, there exist $\epsilon_1(M)$ and $\epsilon_2(M)$ s.t.
   $$\sup_{\xi \in \mathrm{SE}(2), 0 \leq t \leq T} \left| (u^{\epsilon_1} - u^{\epsilon_2})(\xi, t) \right| \geq M(\epsilon_1 - \epsilon_2)^{\alpha},$$

   employing

   $$\omega(\xi, \eta, t) = u^{\epsilon_1}(\xi, t) - u^{\epsilon_2}(\eta, t) - \phi(\xi, \eta, t),$$

   with penalization

   $$\phi(\xi, \eta, t) = \frac{\mu}{\gamma}(\epsilon_1 - \epsilon_2)^{1-\frac{\gamma}{2}} |\xi - \eta|_0^{\gamma} + \frac{M}{2T}(\epsilon_1 - \epsilon_2)^{\alpha} t.$$

Remarks on ω and ϕ

Contradictory hypothesis

$$\sup_{\xi \in SE(2), 0 \le t \le T} \left| (u^{\epsilon_1} - u^{\epsilon_2})(\xi, t) \right| \ge M(\epsilon_1 - \epsilon_2)^\alpha$$

Test function and penalization

- $\omega(\xi, \eta, t) = u^{\epsilon_1}(\xi, t) - u^{\epsilon_2}(\eta, t) - \phi(\xi, \eta, t)$
- $\phi(\xi, \eta, t) = \frac{\mu}{\gamma}(\epsilon_1 - \epsilon_2)^{1-\frac{\gamma}{2}} |\xi - \eta|_0^\gamma + \frac{M}{2T}(\epsilon_1 - \epsilon_2)^\alpha t$

1. Parameters doubled: Derivatives of $|\xi - \eta|_0^\gamma$
2. Penalization with large γ: $|\xi - \eta|_0 \to 0$
3. Attainability of $\sup \omega$: $|\xi| \not\to \infty$ or $|\eta| \not\to \infty$
4. Opposite derivatives: $D_\xi \phi = -D_\eta \phi$
5. Estimates on u^{ϵ_1} and u^{ϵ_2} derivatives at $(\hat{\xi}, \hat{\eta}, \hat{t})$ where $\sup \omega = \omega(\hat{\xi}, \hat{\eta}, \hat{t})$

Conclusions from uniqueness

$$\sup_{\xi \in SE(2), 0 \leq t \leq T} \left| (u^{\epsilon_1} - u^{\epsilon_2})(\xi, t) \right| \leq M(\epsilon_1 - \epsilon_2)^{\alpha}$$

1. $$\sup_{\xi \in SE(2), 0 \leq t \leq T} \left| (u^{\epsilon_1} - u^{\epsilon_2})(\xi, t) \right| \leq M(\epsilon_1 - \epsilon_2)^{\alpha}$$
$$=$$
$$\sup_{\xi \in SE(2), 0 \leq t \leq T} \left| (u^{\epsilon_1} - u)(\xi, t) \right| \leq M(\epsilon_1)^{\alpha} \quad \text{as} \quad \epsilon_2 \to 0$$
$$\Longrightarrow$$
$$u^{\epsilon_1} \to u \quad \text{as} \quad \epsilon_1 \to 0$$

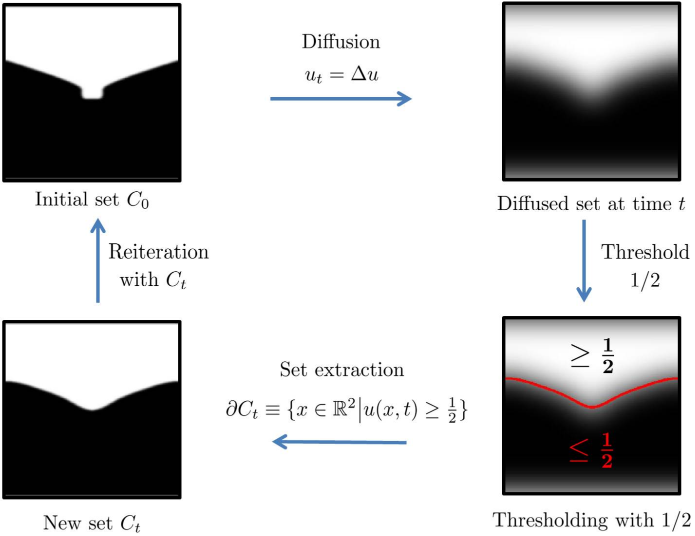
2. Not dependent on u_0
3. Dependence only on $\Gamma_0 = \{\xi \in SE(2) \mid u_0(\xi) = 0\}$

PART II

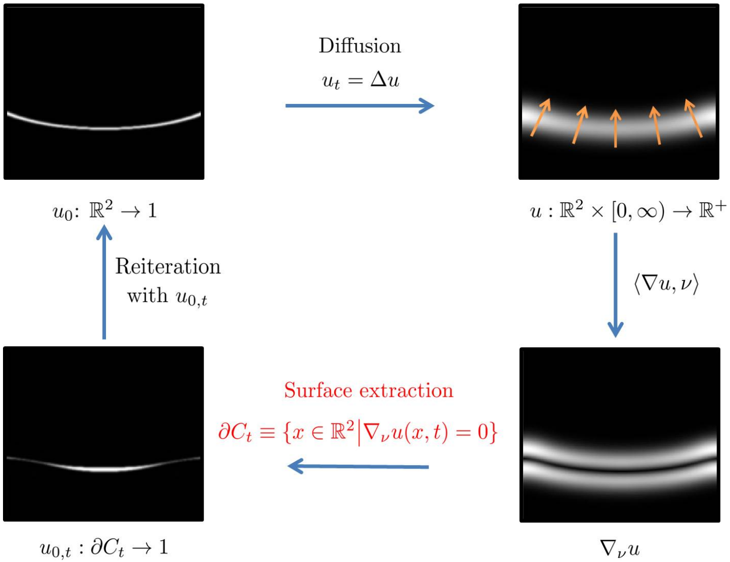
A new diffusion driven motion

Bence-Merriman-Osher algorithm

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = \chi_{C_0} & \text{in } C_0 \times \{t = 0\} \end{cases}$$



Citti-Sarti diffusion driven motion



A new Euclidean diffusion driven motion

$$x = (x_1, \ldots, x_{n-1}, x_n(x_1, \ldots, x_{n-1})) = x_0 + tv\nu \in \mathbb{R}^n$$

New surface definition

$$\partial C_t \equiv \{x \in \mathbb{R}^n \mid \langle \nabla u(x,t), r \rangle = 0\}$$

Gradient along unit normal

$$r = \nu \implies v \approx K \quad \text{as } t \to 0$$

Gradient along fixed direction r

$$v \approx \frac{\langle r, e_n \rangle}{\langle \nu, e_n \rangle \langle \nu, r \rangle} K \quad \text{as } t \to 0$$

A new sub-Riemannian diffusion driven motion

$$\xi = \big(x, y, \theta(x, y)\big) = \xi_0 + tv\nu_h \in \mathrm{SE}(2), \quad X_2 = \partial_\theta$$

New surface definition

$$\partial C_t \equiv \{x \in \mathrm{SE}(2) \mid \langle \nabla_h u(x,t), r \rangle = 0\}$$

Gradient along unit normal

$$r = \nu_h \implies v \approx K_h \quad \text{as } t \to 0$$

Gradient along fixed direction r

$$v \approx \frac{\langle r, X_2 \rangle}{\langle \nu, X_2 \rangle \langle \nu, r \rangle} K_h \quad \text{as } t \to 0$$

Summary and future work

Summary and future work

Main findings

- Uniqueness of vanishing viscosity solutions
- A new diffusion driven motion

Future work

- Implementation of sub-Riemannian mean curvature flow
- Extension to other Lie groups

Thank you!