Sub-Riemannian Problems on Lie Groups with Applications to Medical Image Processing

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The project is devoted to the usage of sub-Riemannian (SR) geodesics on Lie groups in image analysis, and we bring metric analysis techniques to the image processing community. It turned out, that introducing SR geometry on an image (more precisely in its lift to a chosen Lie group) shows very promising results in the problem of extraction of salient curves from the image.

The following problems were studied:

(A) SR length minimizers on SE(2) with a given external cost,

(B) SR geodesics on SO(3) with cuspless spherical projection constraint,

(C) SR geodesics on SE(3) with cuspless spatial projection constraint.

These problems have applications in medical image analysis, namely solutions to (A) and (B) provide a vessel tracking algorithm in flat and spherical images of the retina, and solution to (C) provides a similar algorithm for tracking of elongated structures in 3D images (e.g. MRI images of human brain). The advantage is, that crossing structures on the images are disentangled after lifting to the group, and in contrast to Riemannian geodesics the SR geodesics do not present shortcuts when filling the gap in corrupted contours.

The problems under consideration can be written as follows

$$\dot{q} = \sum_{i=1}^{d} u_i X_i(q), \qquad q \in G, \quad (u_1, \ldots, u_d) \in U \subset \mathbb{R}^d, \tag{1}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{2}$$

$$\int_0^{t_1} C(q) \sqrt{\xi^2 u_1^2 + \sum_{i=2}^{d} u_i^2}\, dt \to \min. \tag{3}$$

Here G is a Lie group; $X_i(q)$ are basis left-invariant vector fields on G, i.e. $X_i(q) = qA_i$, where A_i form a basis of Lie algebra $\mathfrak{g} = T_e G = \text{span}(\, A_1, \ldots, A_n\,)$; and $C : G \to [\delta, 1]$, $\delta > 0$ is a smooth function that we call "given external cost".

The problem (A) is given by (1)–(3) with $G = \text{SE}(2)$, $d = 2$, and $U = \mathbb{R}^2$. Here the external cost C is computed from image data. We present a new flexible wavefront propagation based algorithm, that consists of a first step where a SR distance map is computed as a viscosity solution of a Hamilton–Jacobi–Bellman (HJB) system derived via Pontryagin maximum principle (PMP). Subsequent backward integration, again relying on PMP, gives the SR minimizers. Comparison with exact solutions in the case $C = 1$ shows a remarkable accuracy

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of the SR spheres and the minimizers. We present numerical computations of Maxwell points and cusp points, which we again verify for the case $C = 1$. Regarding image analysis applications, trackings of elongated structures in retinal and synthetic images show that the algorithm generically deals with crossings. (Joint research with E.J.Bekkers, R.Duits and G.R. Sanguinetti).

The problem (B) is given by (1)–(3) with $G = \mathrm{SO}(3)$, $d = 2$, and $U = \mathbb{R}^+ \times \mathbb{R}$. This problem is an extension of a model by J. Petitot, G. Citti, A. Sarti, that takes into account a spherical structure of retina. Furthermore, the data-driven SR minimizers on SO(3) can be used for vessel tracking in spherical images of the retina, in analogy with the SE(2) group and flat images, but now we reduce distortion. In the case $C = 1$ we obtained exact formulas for SR-geodesics, found the first cusp time along the geodesics, and parameterized the set of reachable endpoints. For the case $C \neq 1$, we plan to adapt the algorithm proposed in (A). (Joint research with R. Duits, Y. L. Sachkov and I. Beschastnyi).

The problem (C) is given by (1)-(3) with $G = \mathrm{SE}(3)$, $d = 3$, and $U = \mathbb{R}^+ \times \mathbb{R}^2$. The Lie group SE(3) is used for processing of 3D-images. In the case $C = 1$ we applied PMP, proved Liouville integrability of the Hamiltonian system and obtained exact formulas for SR geodesics. Moreover a numerical investigation of the absence of conjugate points was done, and some results on geometrical properties of geodesics were obtained. For the case $C \neq 1$, we plan to adapt the algorithm proposed in (A).
(Joint research with R. Duits, A. Ghosh and T.C.J. Dela Haije).

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