STABLE AREA-STATIONARY SURFACES IN SUB-RIEMANNIAN GEOMETRY

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Let M be a sub-Riemannian manifold and let Σ a surface immersed in M. We can consider the sub-Riemannian area functional

$$A(\Sigma) = \int_\Sigma |N_h| \, d\Sigma, \tag{0.1}$$

where N denotes the Riemannian unit normal, N_h its projection on the horizontal distribution $\mathcal{H}$ and $d\Sigma$ the Riemannian area element. We observe that an auxiliary Riemannian metric can always been considered on a sub-Riemannian structure, furthermore (0.1) does not depend by the extension used. We call Σ area-stationary if it is a critical point of (0.1). A critical point is stable (resp. strictly stable) when the second variation of the area is always non-negative (resp. positive), for any admissible non-trivial variation of Σ.

When Σ is C^2_h, the first variation formula implies that the mean curvature H of an area-stationary surface vanishes, where

$$H = \operatorname{div}_\Sigma(\nu_h), \tag{0.2}$$

and $\nu_h = N_h/|N_h|$ is the horizontal unit normal. The regularity of C^2_h area-stationary surfaces is well-understood: they are foliated by smooth horizontal curves outside the singular set $\Sigma_0 = \{p \in \Sigma : T_p\Sigma = \mathcal{H}_p\}$, [5]. For surfaces with a priori low regularity, the same result is known for t-graphs in $\mathbb{H}^1$, [3], and for limits of Riemannian minimal graphs in $\mathbb{H}^n$, [1] and [2]. We observe that the case $n = 1$ has a different behaviour w.r.t $n > 1$.

In the recent works [7], [8] and [6] we study the regularity of a prescribed mean curvature surface inside a three-dimensional contact sub-Riemannian manifold, generalizing [3] and [1]. More in details, we have

Theorem 0.1. *Let M be a 3-dimensional contact sub-Riemannian manifold, Ω a domain, and $E \subset \Omega$ a set of prescribed mean curvature $f \in C^k(\Omega)$ with Euclidean Lipschitz boundary $\partial E = \Sigma$. If Σ is locally a Lipschitz regular graph, then the characteristic curves are of class C^{k+2}.*

Furthermore in [7] we study the Bernstein problem, showing that

Theorem 0.2. *Let $\Sigma \subset \mathbb{H}^1$ be a complete oriented stable surface of class C^1 without singular points. Then Σ is a vertical plane.*

As an immediate consequence we have

Corollary 0.3. *Let $\Sigma \subset \mathbb{H}^1$ be a complete locally area-minimizing intrinsic graph of a C^1 function. Then Σ is a vertical plane.*

In higher dimensions we investigate if the stability condition for a minimal graph in $\mathbb{H}^n$, $n > 1$, implies the area-minimizing property. First we construct a linear second order sub-elliptic operator L associated to the second variation of the area

$$A''(\varphi_t(\Sigma))\Big|_{t=0} = -\int_{\Sigma} v\, L(v)\, |N_h|\, d\Sigma,$$

where u is a test function depending on the initial velocity of the variation. Combining interior Schauder estimates of L with a calibration technique we obtain in [4]

Theorem 0.4. *Let $u \in C_h^{2,\alpha}(\tilde{\Omega})$ and let G_u be a minimal graph in $\mathbb{H}^n$, $n > 1$. We assume that G_u is strictly stable in a domain $\Omega \subset \tilde{\Omega}$. Then there exists a tubular neighborhood U of Ω such that for any C_h^1 graph $S \subset U$, $\partial\Omega = \partial S$, we have $A(G_u) \leqslant A(S)$ or $G_u = S$.*

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